

# Classification of Left-Covariant Differential Calculi on the Quantum Group $SL_q(2)^*$

István Heckenberger<sup>†</sup>

Universität Leipzig, Augustusplatz 10-11,  
04109 Leipzig, Germany

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## Abstract

For transcendental values of  $q$  the quantum tangent spaces of all left-covariant first order differential calculi of dimension less than four on the quantum group  $SL_q(2)$  are given. All such differential calculi  $\Gamma$  are determined and investigated for which the left-invariant differential one-forms  $\omega(u_2^1)$ ,  $\omega(u_1^2)$  and  $\omega(u_1^1 - u_2^2)$  generate  $\Gamma$  as a bimodule and the universal higher order differential calculus has the same dimension as in the classical case. Important properties (cohomology spaces,  $*$ -structures, braidings, generalized Lie brackets) of these calculi are examined as well.

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<sup>†</sup>e-mail: heckenbe@mathematik.uni-leipzig.de

# 1 Introduction

The theory of bicovariant differential calculus over Hopf algebras is one of the commonly used and best understood theories related to quantum groups. Its origin was a paper of S. L. Woronowicz [14] where also left-covariant differential calculi were considered.

It is well known that bicovariant differential calculi on quantum groups often (but not always, see [6],[2]) have the unpleasant property (besides non-uniqueness) that their dimensions do not coincide with the dimensions of the canonical differential calculi of the corresponding Lie groups. There were made some attempts using a generalized adjoint action in order to circumvent this defect [3],[12]. Another way is to look for left-covariant differential calculi on the quantum group. Then the corresponding quantum tangent spaces are not invariant under the adjoint action in general. The first such example (the legendary 3D-calculus) was developed by S. L. Woronowicz [13] for the quantum group  $SU_q(2)$ . Among others it was shown therein that the cohomology spaces of the differential complex are the same as in the classical situation. Further examples of such kind were given by K. Schmüdgen and A. Schüler [9, 10]. The paper [9] contains also a first classification of left-covariant differential calculi on the quantum group  $SL_q(2)$  (under very restrictive conditions). A method for the construction of left-covariant differential calculi on a quantum linear group was initiated by K. Schmüdgen [8]. Since the pioneering work of Woronowicz the general theory of differential calculi on quantum groups was refined and developed further. An extensive overview can be found in Chapter 14 of the monograph [7].

In contrast to the classical situation there is no distinguished differential calculus on a quantum group and the non-commutative geometry of a quantum group depends on the differential calculus in general. Thus it seems to be natural to ask how many such calculi (satisfying other additional natural conditions) do really exist. This problem is studied in the present paper. Our aim is to give a step-by-step classification of left-covariant differential calculi  $(\Gamma, d)$  on the quantum group  $SL_q(2)$  under rather general assumptions. In

order to motivate the addition of further assumptions we investigate the outcome of the classification after each step. Although the methods we use can be easily described, the computations are very boring. The computer algebra program FELIX [1] by J. Apel and U. Klaus was very helpful to carry out long computations. The main result of the present paper is Theorem 9. Suppose that  $q$  is a nonzero complex number and not a root of unity. Then the assertion of Theorem 9 states that there are exactly 11 (single) left-covariant first order differential calculi over the Hopf algebra  $\mathcal{O}(\mathrm{SL}_q(2))$  having the following properties: The quantum tangent space is a subspace of the algebra  $\mathcal{U}$  (see Section 3), the one-forms  $\omega(u_2^1)$ ,  $\omega(u_1^2)$  and  $\omega(u_1^1 - u_2^2)$  form a basis of the (necessarily 3-dimensional)  $\mathcal{O}(\mathrm{SL}_q(2))$ -bimodule  $\Gamma$ , the dimension of the space of left-invariant differential 2-forms in the universal higher order differential calculus is at least 3 and the first order calculus is invariant with respect to all Hopf algebra automorphisms of  $\mathcal{O}(\mathrm{SL}_q(2))$ . The list of these calculi is given in Corollary 7. Using the method of Woronowicz [13] it is proved in Theorem 11 that the dimensions of the cohomology spaces of these 11 differential complexes are the same as in the classical situation.

This paper is organized as follows. In Section 2 we recall some basic notions and facts about the general theory of left-covariant differential calculus on quantum groups. If not stated otherwise we follow the definitions and notations of Woronowicz [14] and of the monograph [7]. In Section 3 the structure of the dual Hopf algebra  $\mathcal{U}$  of  $\mathcal{O}(\mathrm{SL}_q(2))$  is described. In Section 4 we determine all 4-dimensional unital right coideals of  $\mathcal{U}$ . In Section 5 further restrictions on the calculus are added. In Section 6 we investigate additional structures such as  $*$ -structures and braidings. In Section 8 the cohomology spaces of the most important differential complexes are studied. Section 7 contains the main theorem (Theorem 9) of the present paper. The outcoming calculi are then studied in detail.

Throughout Sweedler's notation for coproducts and coactions and Einsteins convention of summing over repeated indices are used. The symbols  $\otimes$  and  $\otimes_{\mathcal{A}}$  denote tensor products over the complex numbers and over an

algebra  $\mathcal{A}$ , respectively. All algebras are complex and unital.

## 2 Left-covariant differential calculi on quantum groups

First let us recall some facts of the general theory (see [14], [7]). Let  $\mathcal{A}$  be a Hopf algebra with coproduct  $\Delta$ , counit  $\varepsilon$ , and invertible antipode  $S$ . An  $\mathcal{A}$ -bimodule  $\Gamma$  is called *first order differential calculus* (FODC for short) over  $\mathcal{A}$ , if there is a linear mapping  $d : \mathcal{A} \rightarrow \Gamma$  such that

- $d$  satisfies the Leibniz rule:  $d(ab) = (da)b + adb$  for any  $a, b \in \mathcal{A}$ ,
- $\Gamma = \text{Lin}\{adb \mid a, b \in \mathcal{A}\}$ .

An FODC  $\Gamma$  is called *left-covariant* if there is a linear mapping  $\Delta_L : \Gamma \rightarrow \mathcal{A} \otimes \Gamma$  such that  $\Delta_L(a(db)c) = \Delta(a) \cdot (\text{id} \otimes d)\Delta(b) \cdot \Delta(c)$ , where  $(a \otimes b) \cdot (c \otimes \rho) = ac \otimes b\rho$  and  $(a \otimes \rho) \cdot (b \otimes c) = ab \otimes \rho c$  for any  $a, b, c \in \mathcal{A}$  and  $\rho \in \Gamma$ . Elements  $\rho \in \Gamma$  for which  $\Delta_L(\rho) = 1 \otimes \rho$  are called *left-invariant*. Because of Theorem 2.1 in [14] any left-covariant  $\mathcal{A}$ -bimodule is a free left module and any basis of the vector space  $\Gamma_L$  of left-invariant 1-forms is a free basis of the left (right)  $\mathcal{A}$ -module  $\Gamma$ . The dimension of  $\Gamma_L$  is called the *dimension* of the FODC  $\Gamma$ . In this paper we are dealing only with finite dimensional FODC.

Suppose that  $\Gamma$  is an  $n$ -dimensional first order differential calculus over  $\mathcal{A}$ . Let us fix a basis  $\{\omega_i \mid i = 1, \dots, n\}$  of  $\Gamma_L$ . Then there are functionals  $X_i$ ,  $i = 1, \dots, n$  in the *dual Hopf algebra*  $\mathcal{A}^\circ$  such that the differential  $d$  can be written in the form

$$da = \sum_{i=1}^n a_{(1)} X_i(a_{(2)}) \omega_i, \quad a \in \mathcal{A}. \quad (1)$$

Recall that  $\mathcal{A}^\circ$  is the set of all linear functionals  $f$  on  $\mathcal{A}$  for which there exist functionals  $f_1, \dots, f_N, g_1, \dots, g_N$  on  $\mathcal{A}$  such that  $f(ab) = \sum_{i=1}^N f_i(a)g_i(b)$  for all  $a, b \in \mathcal{A}$ .

The vector space  $\mathcal{X}_\Gamma := \text{Lin}\{X_i \mid i = 1, \dots, n\}$  is called the *quantum tangent space* of the left-covariant FODC  $\Gamma$ . We define a mapping  $\omega : \mathcal{A} \rightarrow \Gamma_L$  by  $\omega(a) = S(a_{(1)})da_{(2)}$ . Then by (1) the equation

$$\omega(a) = \sum_{i=1}^n X_i(a)\omega_i, \quad a \in \mathcal{A} \quad (2)$$

holds. Since  $d1 = 0$ , we have  $\omega(1) = 0$  and  $X_i(1) = 0$  for any  $i$ . The following lemma is the starting point for the first part of our classification.

**Lemma 1.** *If  $\mathcal{X}$  is the quantum tangent space of a FODC  $\Gamma$ , then  $\bar{\mathcal{X}} = \mathcal{X} \oplus \mathbb{C}\varepsilon$  is a unital right coideal of  $\mathcal{A}^\circ$  (i. e.  $\Delta(\bar{\mathcal{X}}) \subset \bar{\mathcal{X}} \otimes \mathcal{A}^\circ$ ). Conversely, any unital right coideal  $\bar{\mathcal{X}}$  of  $\mathcal{A}^\circ$  determines a unique FODC with quantum tangent space  $\mathcal{X}^+ := \{X \in \bar{\mathcal{X}} \mid X(1) = 0\}$ .*

**Proof.** See [7] and [4]. ■

In particular, the coproduct of elements of the quantum tangent space takes the form

$$\Delta X_i = 1 \otimes X_i + X_j \otimes f_i^j, \quad (3)$$

where the functionals  $f_i^j \in \mathcal{A}^\circ$  describe the bimodule structure of  $\Gamma$ :

$$\omega_i a = a_{(1)} f_j^i(a_{(2)}) \omega_j \quad \text{for any } a \in \mathcal{A}. \quad (4)$$

Left-covariant first order differential calculi  $\Gamma$  over  $\mathcal{A}$  are also characterized by the right ideal

$$\mathcal{R}_\Gamma := \{a \in \ker \varepsilon \subset \mathcal{A} \mid \omega(a) = 0\} = \{a \in \mathcal{A} \mid X(a) = 0 \quad \forall X \in \bar{\mathcal{X}}_\Gamma\} \quad (5)$$

of  $\mathcal{A}$ . Two FODC  $(\Gamma_1, d_1)$  and  $(\Gamma_2, d_2)$  over  $\mathcal{A}$  are called *isomorphic*, if  $\mathcal{R}_{\Gamma_1} = \mathcal{R}_{\Gamma_2}$  or equivalently if  $\mathcal{X}_{\Gamma_1} = \mathcal{X}_{\Gamma_2}$ .

Let  $\Gamma^{\otimes k}$  denote the  $k$ -fold tensor product  $\Gamma \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \Gamma$  of the  $\mathcal{A}$ -bimodule  $\Gamma$ ,  $\Gamma^{\otimes 0} := \mathcal{A}$ ,  $\Gamma^{\otimes 1} := \Gamma$  and  $\Gamma^{\otimes} := \bigoplus_{k=0}^{\infty} \Gamma^{\otimes k}$ . Then  $\Gamma^{\otimes}$  becomes an algebra with multiplication  $\otimes_{\mathcal{A}}$ . Let  $\mathcal{S}$  be a graded two-sided ideal in  $\Gamma^{\otimes}$ ,

$\mathcal{S} \subset \bigoplus_{k=2}^{\infty} \Gamma^{\otimes k}$ ,  $\mathcal{S} = \bigoplus_{k=2}^{\infty} \mathcal{S} \cap \Gamma^{\otimes k}$ . Then the  $\mathcal{A}$ -bimodule  $\Gamma^\wedge := \Gamma^\otimes / \mathcal{S}$  as well as  $\Gamma^{\wedge k} := \Gamma^{\otimes k} / (\mathcal{S} \cap \Gamma^{\otimes k})$  are well defined. The bimodule  $\Gamma^\wedge$  is called a *differential calculus* over the Hopf algebra  $\mathcal{A}$  if there is a linear mapping  $d : \Gamma^\wedge \rightarrow \Gamma^\wedge$  of grade one (i.e.  $d : \Gamma^{\wedge k} \rightarrow \Gamma^{\wedge k+1}$ ) such that

- $d$  satisfies the graded Leibniz rule  $d(\rho \wedge \rho') = d\rho \wedge \rho' + (-1)^m \rho \wedge d\rho'$  for  $\rho \in \Gamma^{\wedge m}, \rho' \in \Gamma^\wedge$ ,
- $d^2 = 0$ ,
- $\Gamma = \text{Lin}\{adb \mid a, b \in \mathcal{A}\}$ .

If  $\Gamma$  is left-covariant, then  $\Gamma^\otimes$  is also left-covariant with  $\Delta_L(\rho \otimes_{\mathcal{A}} \rho') = \rho_{(-1)} \rho'_{(-1)} \otimes \rho_{(0)} \otimes_{\mathcal{A}} \rho'_{(0)}$ . Suppose that  $\mathcal{S}$  is an invariant subspace of the left coaction, i.e.  $\Delta_L(\mathcal{S}) \subset \mathcal{A} \otimes \mathcal{S}$ . Then  $\Gamma^\wedge$  inherits the left coaction of  $\Gamma^\otimes$  and  $\Gamma^\wedge$  is called a *left-covariant differential calculus* over  $\mathcal{A}$ .

Suppose that  $\Gamma^\wedge$  is a left-covariant differential calculus over  $\mathcal{A}$ . Then the Maurer-Cartan formula

$$d\omega(a) = -\omega(a_{(1)}) \wedge \omega(a_{(2)}), \quad a \in \mathcal{A} \quad (6)$$

is always fulfilled. Moreover, for any given left-covariant FODC  $\Gamma$  over  $\mathcal{A}$  there exists a universal differential calculus  ${}_u\Gamma^\wedge$ . This means that any left-covariant differential calculus  $\tilde{\Gamma}^\wedge$  over  $\mathcal{A}$  with  $\tilde{\Gamma}^{\wedge 1} = \Gamma$  is isomorphic to  ${}_u\Gamma^\wedge / \tilde{\mathcal{S}}$ , where  $\tilde{\mathcal{S}}$  is a two-sided ideal in  ${}_u\Gamma^\wedge$ . The differential calculus  ${}_u\Gamma^\wedge$  can be given by the two-sided ideal  $\mathcal{S}$  generated by the elements of the vector space

$$\mathcal{S}^2_{\text{L}} := \text{Lin}\{\omega(a_{(1)}) \otimes_{\mathcal{A}} \omega(a_{(2)}) \mid a \in \mathcal{R}\}. \quad (7)$$

**Lemma 2.** *The following equation holds for any left-covariant differential calculus  $\Gamma$  over  $\mathcal{A}$  with quantum tangent space  $\mathcal{X}$ :*

$$\dim({}_u\Gamma^{\wedge 2})_{\text{L}} = \dim\{T \in \bar{\mathcal{X}} \otimes \bar{\mathcal{X}} \mid mT = 0\} - \dim \mathcal{X}, \quad (8)$$

where  $m$  denotes the multiplication map  $m : \bar{\mathcal{X}} \otimes \bar{\mathcal{X}} \subset \mathcal{A}^\circ \otimes \mathcal{A}^\circ \rightarrow \mathcal{A}^\circ$ .

**Proof.** It was proved in [11] that in the present situation the formula

$$\dim({}_u I^{\wedge 2})_{\mathbb{L}} = \dim\{T \in \mathcal{X} \otimes \mathcal{X} \mid \mathfrak{m}T \in \mathcal{X}\} \quad (9)$$

is valid. Let  $V, V'$ , and  $W$  denote the vector spaces

$$V := \{T \in \mathcal{X} \otimes \mathcal{X} \mid \mathfrak{m}T \in \mathcal{X}\}, \quad V' := \{T \in \bar{\mathcal{X}} \otimes \mathcal{X} \mid \mathfrak{m}T = 0\}, \quad \text{and}$$

$$W := \{T \in \bar{\mathcal{X}} \otimes \bar{\mathcal{X}} \mid \mathfrak{m}T = 0\},$$

respectively. Then the mapping  $\varphi : V \rightarrow V'$ ,  $\varphi(T) = T - 1 \otimes \mathfrak{m}T$ , and its inverse  $\psi : V' \rightarrow V$ ,  $\psi(T) = T - (\varepsilon \otimes \text{id})T$  give an isomorphism between  $V$  and  $V'$ . Obviously we have  $\bar{\mathcal{X}} \otimes \bar{\mathcal{X}} = (\bar{\mathcal{X}} \otimes \mathcal{X}) \oplus (\bar{\mathcal{X}} \otimes \mathbb{C} \cdot 1)$  and hence

$$\dim W = \dim V' + \dim W',$$

where  $W' := \{T' \in \bar{\mathcal{X}} \mid \exists T \in \bar{\mathcal{X}} \otimes \mathcal{X}, \mathfrak{m}(T' \otimes 1 - T) = 0\}$ . But the vector space  $W'$  is isomorphic to  $\mathcal{X}$ . Indeed,  $\mathcal{X} \subset W'$  since  $\mathfrak{m}(X \otimes 1 - 1 \otimes X) = 0$  but  $1 \notin W'$ . The latter follows from the fact that  $\varepsilon(\mathfrak{m}(1 \otimes 1)) = 1$  and  $\varepsilon(\mathfrak{m}(T)) = 0$  for any  $T \in \bar{\mathcal{X}} \otimes \mathcal{X}$ .  $\blacksquare$

### 3 The Hopf dual of $\mathcal{O}(\text{SL}_q(2))$

In what follows we assume that  $q$  is a transcendental complex number. The structure of the coordinate Hopf algebra  $\mathcal{O}(\text{SL}_q(2))$  (with generators  $u_j^i$ ,  $i, j = 1, 2$ ) of the quantum group  $\text{SL}_q(2)$  is well known. We now restate the description of the Hopf dual  $\mathcal{U} = \mathcal{O}(\text{SL}_q(2))^\circ$  obtained in the monograph [5]. Let  $\mathcal{U}$  denote the unital algebra generated by the elements  $E, F, G$  and  $f_\mu$  ( $\mu \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ ) and by the relations

$$\begin{aligned} f_\mu f_\nu &= f_{\mu\nu}, & f_\mu E &= \mu^2 E f_\mu, & f_\mu F &= \mu^{-2} F f_\mu, & f_\mu G &= G f_\mu, \\ GE &= E(G + 2), & GF &= F(G - 2), & EF - FE &= \frac{f_q - f_{q^{-1}}}{q - q^{-1}}. \end{aligned} \quad (10)$$

The element  $f_1$  is the unit in the algebra  $\mathcal{U}$ . The element  $f_{-1}$  is also denoted by  $\varepsilon_-$ .

Let us fix one square root  $q^{1/2}$  of  $q$  and define  $K = f_{q^{1/2}}$ . Then there is a Hopf algebra structure on  $\mathcal{U}$  such that

$$\begin{aligned}\Delta(E) &= E \otimes K + K^{-1} \otimes E, & \varepsilon(E) &= 0, & S(E) &= -qE, \\ \Delta(F) &= F \otimes K + K^{-1} \otimes F, & \varepsilon(F) &= 0, & S(F) &= -q^{-1}F, \\ \Delta(G) &= 1 \otimes G + G \otimes 1, & \varepsilon(G) &= 0, & S(G) &= -G, \\ \Delta(f_\mu) &= f_\mu \otimes f_\mu, & \varepsilon(f_\mu) &= 1, & S(f_\mu) &= f_{\mu^{-1}}.\end{aligned}\tag{11}$$

To make calculations easier we use the notation  $F^{(k)} := F^k K^{-k} / [k]!$ ,  $E^{(k)} := K^{-k} E^k / [k]!$  and  $G^{(k)} = G^k / k!$  for any  $k \in \mathbb{N}_0$ , where  $[k]! = [k][k-1] \cdots [1]$ ,  $[0]! = 1$  and  $[k] = (q^k - q^{-k}) / (q - q^{-1})$ . The coproducts of these elements are

$$\begin{aligned}\Delta(F^{(k)}) &= \sum_{r=0}^k F^{(k-r)} K^{-2r} \otimes F^{(r)}, \\ \Delta(E^{(k)}) &= \sum_{s=0}^k K^{-2s} E^{(k-s)} \otimes E^{(s)}, \\ \Delta(G^{(k)}) &= \sum_{t=0}^k G^{(k-t)} \otimes G^{(t)}.\end{aligned}\tag{12}$$

The dual pairing  $\langle \cdot, \cdot \rangle$  of the Hopf algebras  $\mathcal{U} = \mathcal{O}(\mathrm{SL}_q(2))^\circ$  and  $\mathcal{O}(\mathrm{SL}_q(2))$  is given by the matrices  $\langle \cdot, u_j^i \rangle$ , where

$$E = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad f_\mu = \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{pmatrix}.\tag{13}$$

For the algebra  $\mathcal{U}$  a PBW-like theorem holds: the elements of the set  $\{F^{(i)} f_\mu E^{(j)} G^{(k)} \mid i, j, k \in \mathbb{N}_0, \mu \in \mathbb{C}^\times\}$  form a vector space basis of the algebra  $\mathcal{U}$ .

**Remark.** That the Hopf algebra  $\mathcal{U}$  defined above is the full Hopf dual of the Hopf algebra  $\mathcal{O}(\mathrm{SL}_q(2))$  is proved in the monograph [5] for transcendental

$q$ . The assumption that  $q$  is transcendental is only necessary in order to apply this theorem. If this fact holds already for  $q$  not a root of unity, then all results of the present paper remain valid under this assumption. In what follows, we classify the corresponding left-covariant differential calculi whose quantum tangent space are contained in  $\mathcal{U}$ . All these considerations hold if  $q \neq 0$  and  $q$  is not a root of unity. ■

## 4 Unital right coideals of $\mathcal{U}$

By Lemma 1 left-covariant first order differential calculi over the Hopf algebra  $\mathcal{A} := \mathcal{O}(\mathrm{SL}_q(2))$  and unital right coideals of  $\mathcal{A}^\circ$  are in one-to-one correspondence. Now we determine all unital right coideals of  $\mathcal{U}$  of dimension  $\leq 4$ . In this way we describe all left-covariant first order differential calculi over  $\mathcal{A}$  of dimension less than four.

For  $X = F^{(i)} f_\mu E^{(j)} G^{(k)}$ ,  $i, j, k \in \mathbb{N}_0$ ,  $\mu \in \mathbb{C}^\times$  we set

$$\partial_1(X) := i, \quad \partial_2(X) := \mu, \quad \partial_3(X) := j, \quad \partial_4(X) := k, \quad \partial_{13}(X) := i + j.$$

For a finite linear combination  $X = \sum_{i=1}^n a_i X_i$  of such elements we define  $\partial_m(X) := \max\{\partial_m(X_i) \mid i = 1, \dots, n\}$  for  $m \in \{1, 3, 4, 13\}$ . Then from the PBW-theorem we conclude that  $\mathcal{U} = \bigoplus_{\mu \in \mathbb{C}^\times} [\mathcal{U}]_\mu$ , where

$$[\mathcal{U}]_\mu := \mathrm{Lin}\{F^{(i)} f_\mu E^{(j)} G^{(k)} \mid i, j, k \in \mathbb{N}_0\}.$$

**Proposition 3.** *Let  $\mathcal{X}$  be a right coideal of  $\mathcal{U}$  and let  $X \in \mathcal{X}$ . Then there exist complex numbers  $\alpha_{i,\nu,j,k}$  ( $\nu \in \mathbb{C}^\times$ ,  $i, j, k \in \mathbb{N}_0$ ) such that  $X = \sum_{i,\nu,j,k} \alpha_{i,\nu,j,k} F^{(i)} f_\nu E^{(j)} G^{(k)}$ . Let us fix  $\mu \in \mathbb{C}^\times$  and  $r, s, t \in \mathbb{N}_0$ . Consider the element*

$$X_{r,\mu,s,t} := \sum_{i,j,k} \alpha_{i,\mu,j,k} F^{(i-r)} f_{q^{-r-s}\mu} E^{(j-s)} G^{(k-t)} \quad (14)$$

*of  $\mathcal{U}$ , where the sum is running over all  $i, j, k \in \mathbb{N}_0$  with  $i \geq r$ ,  $j \geq s$  and  $k \geq t$ . Then the vector space  $\mathrm{Lin}\{X_{r\mu st} \mid r, s, t \in \mathbb{N}_0, \mu \in \mathbb{C}^\times\}$  is the smallest (with respect to inclusion) right coideal of  $\mathcal{U}$  containing  $X$ .*

**Proof.** Using (12) we compute the coproduct of  $X$  and obtain

$$\begin{aligned}
\Delta \left( \sum_{i\nu jk} \alpha_{i\nu jk} F^{(i)} f_\nu E^{(j)} G^{(k)} \right) &= \\
&= \sum_{i\nu jk} \alpha_{i\nu jk} \sum_{rst} F^{(i-r)} K^{-2r} f_\nu K^{-2s} E^{(j-s)} G^{(k-t)} \otimes F^{(r)} f_\nu E^{(s)} G^{(t)} \\
&= \sum_{rvst} \sum_{ijk} \alpha_{i\nu jk} F^{(i-r)} f_{q^{-r-s}\nu} E^{(j-s)} G^{(k-t)} \otimes F^{(r)} f_\nu E^{(s)} G^{(t)} \\
&= \sum_{rvst} X_{r\nu st} \otimes F^{(r)} f_\nu E^{(s)} G^{(t)}.
\end{aligned}$$

Because of the PBW-theorem the elements of the right hand side of the tensor product are linearly independent. Hence the elements  $X_{r\nu st}$  belong to any right coideal containing  $X$  (i.e. they belong to  $\mathcal{X}$  too). Observe that  $X = \sum_{\mu \in \mathbb{C}^\times} X_{0\mu 00}$ . Taking in account that  $(\Delta \otimes \text{id})\Delta(X) = (\text{id} \otimes \Delta)\Delta(X)$  it follows that the vector space  $\text{Lin}\{X_{r\mu st} \mid r, s, t \in \mathbb{N}_0, \mu \in \mathbb{C}^\times\}$  is a right coideal of  $\mathcal{U}$ .  $\blacksquare$

Since  $X = \sum_{\mu \in \mathbb{C}^\times} X_{0\mu 00}$  we obtain the following.

**Corollary 4.** *Any right coideal  $\mathcal{X}$  of  $\mathcal{U}$  is isomorphic to the direct sum  $\bigoplus_{\mu \in \mathbb{C}^\times} [\mathcal{X}]_\mu$  of its homogeneous components  $[\mathcal{X}]_\mu := \mathcal{X} \cap [\mathcal{U}]_\mu$ .*

What can be said about the dimension of a right coideal  $\mathcal{X}$  of  $\mathcal{U}$ ? Let  $X$  be a nonzero element of  $\mathcal{X}$ . By Corollary 4 we may assume without loss of generality that there is a  $\mu \in \mathbb{C}^\times$  such that  $X \in [\mathcal{X}]_\mu$ . Due to the PBW-theorem there is for any  $t \in \mathbb{N}_0$ ,  $t \leq \partial_4(X)$  a unique  $X_t \in \mathcal{U}$  such that  $X = \sum_{t=0}^{\partial_4(X)} X_t G^{(t)}$  and  $X_{\partial_4(X)} \neq 0$ . Let now  $p \in \mathbb{N}_0$ ,  $p \leq \partial_{13}(X)$ . We define

$$t_{13}(p) = \max\{m \in \mathbb{Z} \mid X_m \neq 0, \partial_{13}(X_m) \geq p\}.$$

Then the coefficients of  $X_{r\mu st}$  in Proposition 3 indicate that for any  $t \in \mathbb{N}_0$ ,  $t \leq t_{13}(p)$  there is at least one number  $r(t, p) \in \mathbb{N}_0$ ,  $r(t, p) \leq p$  such that  $X_{r(t, p), \mu, p-r(t, p), t} \neq 0$ . Since  $\partial_2(X_{r(t, p), \mu, p-r(t, p), t}) = \mu q^{-p}$ ,  $q$  is not a root of unity and  $\partial_4(X_{r(t, p), \mu, p-r(t, p), t}) = t_{13}(p) - t$  we conclude that the elements

$X_{r(t,p),\mu,p-r(t,p),t}$ ,  $0 \leq p \leq \partial_{13}(X)$ ,  $0 \leq t \leq t_{13}(p)$  are linearly independent. Hence the number

$$\sum_{p=0}^{\partial_{13}(X)} (t_{13}(p) + 1) = \sum_{t=0}^{\partial_4(X)} (s_{13}(t) + 1) \quad (15)$$

with  $s_{13}(t) = \max\{\partial_{13}(X_m) \mid m \geq t, X_m \neq 0\}$  for  $t \in \mathbb{N}_0$ ,  $t \leq \partial_4(X)$ , is a lower bound for the dimension of the right coideal  $\mathcal{X}$ .

**Example 1.** Suppose that  $X = K^6 G^{(5)} + (F^{(1)} K^6 - K^6) G^{(4)} + (F^{(2)} K^6 E^{(1)} + K^6 E^{(3)}) G^{(2)} + F^{(3)} K^6$  is an element of a right coideal  $\mathcal{X}$  of  $\mathcal{U}$ . We have  $X \in [\mathcal{U}]_{q^3}$ ,  $\partial_4(X) = 5$  and  $\partial_{13}(X) = 3$ . The coefficients  $X_k$  of  $G^{(k)}$  in  $X$  are  $X_0 = F^{(3)} K^6$ ,  $X_2 = F^{(2)} K^6 E^{(1)} + K^6 E^{(3)}$ ,  $X_4 = F^{(1)} K^6 - K^6$ ,  $X_5 = K^6$  and  $X_t = 0$  otherwise. The values of the functions  $t_{13}(n)$  ( $s_{13}(n)$ ) are 5, 4, 2, 2 for  $n = 0, 1, 2, 3$  (3, 3, 3, 1, 1, 0 for  $n = 0, 1, 2, 3, 4, 5$ ). Hence the dimension of each right coideal  $\mathcal{X}$  containing  $X$  is at least 17. ■

In the remaining part of this section we determine all unital right coideals of  $\mathcal{U}$  of dimension  $\leq 4$ .

#### 4.1 $\dim \mathcal{X} \leq 2$

The following list contains all possibilities (if not otherwise stated, parameters are arbitrary complex numbers):

- $\mathcal{X}_1^1 = \mathbb{C} \cdot 1$ .
- $\mathcal{X}_1^2 = \text{Lin}\{G, 1\}$ ,
- $\mathcal{X}_2^2 = \text{Lin}\{F^{(1)} K^2 + \alpha K^2 E^{(1)} + \beta K^2, 1\}$ ,
- $\mathcal{X}_3^2 = \text{Lin}\{K^2 E^{(1)} + \alpha K^2, 1\}$ ,
- $\mathcal{X}_4^2 = \text{Lin}\{f_\mu, 1\}$ ,  $\mu \in \mathbb{C}^\times$ ,  $\mu \neq 1$ .

Obviously  $\dim \mathcal{X} = 1$  and  $1 \in \mathcal{X}$  imply that the only 1-dimensional unital right coideal of  $\mathcal{U}$  is  $\mathcal{X}_1^1$ .

Suppose that  $\dim \mathcal{X} = 2$ . By (15) we have  $\partial_4(X) < \dim \mathcal{X} = 2$  for any  $X \in \mathcal{X}$ . If there is an  $X \in [\mathcal{X}]_\mu$ ,  $\mu \in \mathbb{C}^\times$  with  $\partial_4(X) = 1$  then we must have  $s_{13}(0) = s_{13}(1) = 0$  by (15) and we obtain  $X = \alpha f_\mu G + \beta f_\mu$ ,  $\alpha \neq 0$ . The only nonzero elements  $X_{rvst}$  are  $X$  and  $X_{0\mu 01} = \alpha f_\mu$ . Since  $\dim \mathcal{X} = 2$  and  $1 \in \mathcal{X}$ , 1 must be one of those two elements. Therefore,  $f_\mu = 1$ , i. e.  $\mu = 1$ . Hence  $\mathcal{X}$  is isomorphic to  $\mathcal{X}_1^2$ .

If  $\partial_4(X) = 0$  for any  $X \in \mathcal{X}$  then by (15)  $s_{13}(0)$  can take the values 1 and 0. In the first case we have  $1 = s_{13}(0) = \partial_{13}(X)$  from which  $X = \alpha F^{(1)} f_\mu + \beta E^{(1)} f_\mu + \gamma f_\mu$  ( $\alpha \neq 0$  or  $\beta \neq 0$ ) follows. Then  $X_{1\mu 00} = \alpha K^{-2} f_\mu$ ,  $X_{0\mu 01} = \beta K^{-2} f_\mu$  and therefore  $K^{-2} f_\mu \in \mathcal{X}$ . Since  $1 \in \mathcal{X}$ , we must have  $f_\mu = K^2$ . This gives the coideals  $\mathcal{X}_2^2$  and  $\mathcal{X}_3^2$ . In the remaining case we have  $\partial_{13}(X) = 0$  and we get  $\mathcal{X}_4^2$ .

## 4.2 $\dim \mathcal{X} = 3$

By (15),  $\partial_4(X) \leq 2$  for any  $X \in \mathcal{X}$ . If  $\partial_4(X) = 2$  then  $s_{13}(i) = 0$  for any  $i = 0, 1, 2$ . If  $\partial_4(X) = 1$  then  $s_{13}(1) = 0$  and  $s_{13}(0)$  can take the values 0 and 1. Finally, if  $\partial_4(X) = 0$  then  $s_{13}(0)$  can be 2, 1 or 0. We obtain the following list of unital right coideals:

- $\mathcal{X}_1^3 = \text{Lin}\{G^2, G, 1\}$ ,
- $\mathcal{X}_2^3 = \text{Lin}\{K^2 G + \alpha F^{(1)} K^2 + \beta K^2 E^{(1)}, K^2, 1\}$ ,  $|\alpha| + |\beta| \neq 0$ ,
- $\mathcal{X}_3^3 = \text{Lin}\{G + \alpha F^{(1)} + \beta E^{(1)}, K^{-2}, 1\}$ ,  $|\alpha| + |\beta| \neq 0$ ,
- $\mathcal{X}_4^3 = \text{Lin}\{f_\mu G, f_\mu, 1\}$ ,  $\mu \in \mathbb{C}^\times$ ,  $\mu \neq 1$ ,
- $\mathcal{X}_5^3 = \text{Lin}\{G, F^{(1)} K^2 + \alpha K^2 E^{(1)} + \beta K^2, 1\}$ ,
- $\mathcal{X}_6^3 = \text{Lin}\{G, K^2 E^{(1)} + \alpha K^2, 1\}$ ,
- $\mathcal{X}_7^3 = \text{Lin}\{G, f_\mu, 1\}$ ,  $\mu \in \mathbb{C}^\times$ ,  $\mu \neq 1$ ,
- $\mathcal{X}_8^3 = \text{Lin}\{F^{(2)} K^4 + \alpha F^{(1)} K^4 E^{(1)} + \alpha^2 K^4 E^{(2)} + \beta F^{(1)} K^4 + \alpha \beta K^4 E^{(1)} + \gamma K^4, F^{(1)} K^2 + \alpha K^2 E^{(1)} + \beta K^2, 1\}$ ,

- $\mathcal{X}_9^3 = \text{Lin}\{K^4E^{(2)} + \alpha K^4E^{(1)} + \beta K^4, K^2E^{(1)} + \alpha K^2, 1\},$
- $\mathcal{X}_{10}^3 = \text{Lin}\{F^{(1)}f_\mu + \alpha f_\mu E^{(1)} + \beta f_\mu, f_{\mu q^{-1}}, 1\}, \mu \neq q,$
- $\mathcal{X}_{11}^3 = \text{Lin}\{f_\mu E^{(1)} + \alpha f_\mu, f_{\mu q^{-1}}, 1\}, \mu \neq q,$
- $\mathcal{X}_{12}^3 = \text{Lin}\{F^{(1)}K^2 + \alpha K^2, K^2E^{(1)} + \beta K^2, 1\},$
- $\mathcal{X}_{13}^3 = \text{Lin}\{F^{(1)}K^2 + \alpha K^2E^{(1)} + \beta K^2, f_\mu, 1\}, \mu \neq 1,$
- $\mathcal{X}_{14}^3 = \text{Lin}\{K^2E^{(1)} + \alpha K^2, f_\mu, 1\}, \mu \neq 1,$
- $\mathcal{X}_{15}^3 = \text{Lin}\{f_\mu, f_\nu, 1\}, \mu \neq \nu, \mu \neq 1, \nu \neq 1.$

### 4.3 $\dim \mathcal{X} = 4$

Finally we give the complete list of 4-dimensional unital right coideals of  $\mathcal{U}$ . For the proof of the completeness of the list the method explained above is used.

- $\mathcal{X}_1^4 = \text{Lin}\{G^{(3)}, G^{(2)}, G, 1\},$
- $\mathcal{X}_2^4 = \text{Lin}\{K^2G^{(2)} + \alpha F^{(1)}K^2 + \beta K^2E^{(1)}, K^2G, K^2, 1\}, |\alpha| + |\beta| \neq 0,$
- $\mathcal{X}_3^4 = \text{Lin}\{G^{(2)} + \alpha F^{(1)} + \beta E^{(1)}, G, K^{-2}, 1\}, |\alpha| + |\beta| \neq 0,$
- $\mathcal{X}_4^4 = \text{Lin}\{f_\mu G^{(2)}, f_\mu G, f_\mu, 1\}, \mu \neq 1,$
- $\mathcal{X}_5^4 = \text{Lin}\{G^{(2)}, G, F^{(1)}K^2 + \alpha K^2E^{(1)} + \beta K^2, 1\},$
- $\mathcal{X}_6^4 = \text{Lin}\{G^{(2)}, G, K^2E^{(1)} + \alpha K^2, 1\},$
- $\mathcal{X}_7^4 = \text{Lin}\{G^{(2)}, G, f_\mu, 1\}, \mu \neq 1,$
- $\mathcal{X}_8^4 = \text{Lin}\{F^{(1)}K^2G + \alpha K^2E^{(1)}G + \beta K^2G + \gamma K^2E^{(1)} + \delta K^2, G, F^{(1)}K^2 + \alpha K^2E^{(1)} + \beta K^2, 1\},$   
(LI)  $\gamma \neq 2\alpha$

- $\mathcal{X}_9^4 = \text{Lin}\{K^2 E^{(1)} G + \alpha K^2 G + \beta F^{(1)} K^2 + \gamma K^2, G, K^2 E^{(1)} + \alpha K^2, 1\},$   
(LI)  $\beta \neq 0$
- $\mathcal{X}_{10}^4 = \text{Lin}\{K^4 G + \alpha F^{(2)} K^4 + \alpha \beta F^{(1)} K^4 E^{(1)} + \alpha \beta^2 K^4 E^{(2)} + \alpha \gamma F^{(1)} K^4 + \alpha \beta \gamma K^4 E^{(1)}, F^{(1)} K^2 + \beta K^2 E^{(1)} + \gamma K^2, K^4, 1\}, \alpha \neq 0,$
- $\mathcal{X}_{11}^4 = \text{Lin}\{K^4 G + \alpha K^4 E^{(2)} + \alpha \beta K^4 E^{(1)}, K^2 E^{(1)} + \beta K^2, K^4, 1\}, \alpha \neq 0,$
- $\mathcal{X}_{12}^4 = \text{Lin}\{G + \alpha F^{(2)} + \alpha \beta F^{(1)} E^{(1)} + \alpha \beta^2 E^{(2)} + \alpha \gamma F^{(1)} + \alpha \beta \gamma E^{(1)}, F^{(1)} K^{-2} + \beta K^{-2} E^{(1)} + \gamma K^{-2}, K^{-4}, 1\}, \alpha \neq 0,$
- $\mathcal{X}_{13}^4 = \text{Lin}\{G + \alpha E^{(2)} + \alpha \beta E^{(1)}, K^{-2} E^{(1)} + \beta K^{-2}, K^{-4}, 1\}, \alpha \neq 0,$
- $\mathcal{X}_{14}^4 = \text{Lin}\{f_\mu G + \alpha F^{(1)} f_\mu + \beta f_\mu E^{(1)}, f_\mu, f_{\mu q^{-1}}, 1\},$   
 $\mu \neq 1, \mu \neq q, |\alpha| + |\beta| \neq 0,$
- $\mathcal{X}_{15}^4 = \text{Lin}\{K^2 G + \alpha F^{(1)} K^2 + \beta K^2 E^{(1)}, G, K^2, 1\}, |\alpha| + |\beta| \neq 0,$
- $\mathcal{X}_{16}^4 = \text{Lin}\{K^2 G + \alpha F^{(1)} K^2 + \beta K^2 E^{(1)}, F^{(1)} K^4 + \gamma K^4 E^{(1)} + \delta K^4, K^2, 1\},$   
 $|\alpha| + |\beta| \neq 0,$   
(LI)  $\beta \neq \alpha \gamma,$
- $\mathcal{X}_{17}^4 = \text{Lin}\{K^2 G + \alpha F^{(1)} K^2 + \beta K^2 E^{(1)}, K^4 E^{(1)} + \gamma K^4, K^2, 1\},$   
 $|\alpha| + |\beta| \neq 0,$   
(LI)  $\alpha \neq 0,$
- $\mathcal{X}_{18}^4 = \text{Lin}\{K^2 G + \alpha K^2 E^{(1)}, F^{(1)} K^2 + \beta K^2 E^{(1)}, K^2, 1\}, \alpha \neq 0,$   
(LI)
- $\mathcal{X}_{19}^4 = \text{Lin}\{K^2 G + \alpha F^{(1)} K^2, K^2 E^{(1)}, K^2, 1\}, \alpha \neq 0,$   
(LI)
- $\mathcal{X}_{20}^4 = \text{Lin}\{K^2 G + \alpha F^{(1)} K^2 + \beta K^2 E^{(1)}, f_\mu, K^2, 1\}, |\alpha| + |\beta| \neq 0, \mu \neq 1,$   
 $\mu \neq q,$
- $\mathcal{X}_{21}^4 = \text{Lin}\{G + \alpha F^{(1)} + \beta E^{(1)}, K^{-2} G, K^{-2}, 1\}, |\alpha| + |\beta| \neq 0,$

- $\mathcal{X}_{22}^4 = \text{Lin}\{G + \alpha E^{(1)}, F^{(1)} + \beta E^{(1)}, K^{-2}, 1\}, \alpha \neq 0,$   
(LI)
- $\mathcal{X}_{23}^4 = \text{Lin}\{G + \alpha F^{(1)}, E^{(1)}, K^{-2}, 1\}, \alpha \neq 0,$   
(LI)
- $\mathcal{X}_{24}^4 = \text{Lin}\{G + \alpha F^{(1)} + \beta E^{(1)}, F^{(1)}K^2 + \gamma K^2 E^{(1)} + \delta K^2, K^{-2}, 1\},$   
 $|\alpha| + |\beta| \neq 0,$   
(LI)  $\beta \neq \alpha\gamma$
- $\mathcal{X}_{25}^4 = \text{Lin}\{G + \alpha F^{(1)} + \beta E^{(1)}, K^2 E^{(1)} + \gamma K^2, K^{-2}, 1\}, |\alpha| + |\beta| \neq 0,$   
(LI)  $\alpha \neq 0$
- $\mathcal{X}_{26}^4 = \text{Lin}\{G + \alpha F^{(1)} + \beta E^{(1)}, f_\mu, K^{-2}, 1\}, \mu \neq 1, \mu \neq q^{-1},$
- $\mathcal{X}_{27}^4 = \text{Lin}\{f_\mu G, G, f_\mu, 1\}, \mu \neq 1,$
- $\mathcal{X}_{28}^4 = \text{Lin}\{f_\mu G, F^{(1)}f_{\mu q} + \alpha f_{\mu q} E^{(1)} + \beta f_{\mu q}, f_\mu, 1\}, \mu \neq 1,$
- $\mathcal{X}_{29}^4 = \text{Lin}\{f_\mu G, f_{\mu q} E^{(1)} + \alpha f_{\mu q}, f_\mu, 1\}, \mu \neq 1,$
- $\mathcal{X}_{30}^4 = \text{Lin}\{f_\mu G, F^{(1)}K^2 + \alpha K^2 E^{(1)} + \beta K^2, f_\mu, 1\}, \mu \neq 1,$
- $\mathcal{X}_{31}^4 = \text{Lin}\{f_\mu G, K^2 E^{(1)} + \alpha K^2, f_\mu, 1\}, \mu \neq 1,$
- $\mathcal{X}_{32}^4 = \text{Lin}\{f_\mu G, f_\mu, f_\nu, 1\}, \mu \neq \nu, \mu \neq 1, \nu \neq 1,$
- $\mathcal{X}_{33}^4 = \text{Lin}\{G, F^{(2)}K^4 + \alpha F^{(1)}K^4 E^{(1)} + \alpha^2 K^4 E^{(2)} + \beta F^{(1)}K^4 + \alpha\beta K^4 E^{(1)} +$   
 $\gamma K^4, F^{(1)}K^2 + \alpha K^2 E^{(1)} + \beta K^2, 1\}, \alpha \neq 0,$
- $\mathcal{X}_{34}^4 = \text{Lin}\{G, K^4 E^{(2)} + \alpha K^4 E^{(1)} + \beta K^4, K^2 E^{(1)} + \alpha K^2, 1\},$
- $\mathcal{X}_{35}^4 = \text{Lin}\{G, F^{(1)}f_\mu + \alpha f_\mu E^{(1)} + \beta f_\mu, f_{\mu q^{-1}}, 1\}, \mu \neq q,$
- $\mathcal{X}_{36}^4 = \text{Lin}\{G, f_\mu E^{(1)} + \alpha f_\mu, f_{\mu q^{-1}}, 1\}, \mu \neq q,$
- $\mathcal{X}_{37}^4 = \text{Lin}\{G, F^{(1)}K^2 + \alpha K^2, K^2 E^{(1)} + \beta K^2, 1\},$   
(LI)

- $\mathcal{X}_{38}^4 = \text{Lin}\{G, F^{(1)}K^2 + \alpha K^2 E^{(1)} + \beta K^2, f_\mu, 1\}, \mu \neq 1,$
- $\mathcal{X}_{39}^4 = \text{Lin}\{G, K^2 E^{(1)} + \alpha K^2, f_\mu, 1\}, \mu \neq 1,$
- $\mathcal{X}_{40}^4 = \text{Lin}\{G, f_\mu, f_\nu, 1\}, \mu \neq \nu, \mu \neq 1, \nu \neq 1,$
- $\mathcal{X}_{41}^4 = \text{Lin}\{F^{(3)}K^6 + \alpha F^{(2)}K^6 E^{(1)} + \alpha^2 F^{(1)}K^6 E^{(2)} + \alpha^3 K^6 E^{(3)} + \beta F^{(2)}K^6 + \alpha\beta F^{(1)}K^6 E^{(1)} + \alpha^2\beta K^6 E^{(2)} + \gamma F^{(1)}K^6 + \alpha\gamma K^6 E^{(1)} + \delta K^6, F^{(2)}K^4 + \alpha F^{(1)}K^4 E^{(1)} + \alpha^2 K^4 E^{(2)} + \beta F^{(1)}K^4 + \alpha\beta K^4 E^{(1)} + \gamma K^4, F^{(1)}K^2 + \alpha K^2 E^{(1)} + \beta K^2, 1\},$
- $\mathcal{X}_{42}^4 = \text{Lin}\{K^6 E^{(3)} + \alpha K^6 E^{(2)} + \beta K^6 E^{(1)} + \gamma K^6, K^4 E^{(2)} + \alpha K^4 E^{(1)} + \beta K^4, K^2 E^{(1)} + \alpha K^2, 1\},$
- $\mathcal{X}_{43}^4 = \text{Lin}\{F^{(2)}f_\mu + \alpha F^{(1)}f_\mu E^{(1)} + \alpha^2 f_\mu E^{(2)} + \beta F^{(1)}f_\mu + \alpha\beta f_\mu E^{(1)} + \gamma f_\mu, F^{(1)}f_{\mu q^{-1}} + \alpha f_{\mu q^{-1}} E^{(1)} + \beta f_{\mu q^{-1}}, f_{\mu q^{-2}}, 1\}, \mu \neq q^2,$
- $\mathcal{X}_{44}^4 = \text{Lin}\{f_\mu E^{(2)} + \alpha f_\mu E^{(1)} + \beta f_\mu, f_{\mu q^{-1}} E^{(1)} + \alpha f_{\mu q^{-1}}, f_{\mu q^{-2}}, 1\}, \mu \neq q^2,$
- $\mathcal{X}_{45}^4 = \text{Lin}\{F^{(2)}K^4 + \alpha_1 F^{(1)}K^4 E^{(1)} + \alpha_2 K^4 E^{(2)} + (\beta_1 + \alpha_1\beta_2)F^{(1)}K^4 + (\alpha_1\beta_1 + \alpha_2\beta_2)K^4 E^{(1)} + \gamma K^4, F^{(1)}K^2 + \beta_1 K^2, K^2 E^{(1)} + \beta_2 K^2, 1\}, \alpha_2 \neq \alpha_1^2, \text{(LI)} (q^2 - q^{-2})\gamma \neq q\alpha_1 + (q^2 - 1)(\beta_1^2 + 2\alpha_1\beta_1\beta_2 + \alpha_2\beta_2^2)$
- $\mathcal{X}_{46}^4 = \text{Lin}\{F^{(2)}K^4 + \alpha F^{(1)}K^4 E^{(1)} + \alpha^2 K^4 E^{(2)} + \beta F^{(1)}K^4 + \gamma K^4 E^{(1)} + \delta K^4, F^{(1)}K^2 + \alpha K^2 E^{(1)}, K^2, 1\}, \gamma \neq \alpha\beta, \text{(LI)}$
- $\mathcal{X}_{47}^4 = \text{Lin}\{F^{(1)}K^4 E^{(1)} + \alpha K^4 E^{(2)} + \beta F^{(1)}K^4 + (\gamma + \alpha\beta)K^4 E^{(1)} + \delta K^4, F^{(1)}K^2 + \gamma K^2, K^2 E^{(1)} + \beta K^2, 1\}, \text{(LI)} (q^2 - q^{-2})\delta \neq q + (q^2 - 1)\beta(2\gamma + \alpha\beta)$
- $\mathcal{X}_{48}^4 = \text{Lin}\{K^4 E^{(2)} + \alpha F^{(1)}K^4 + \beta K^4 E^{(1)} + \gamma K^4, K^2 E^{(1)}, K^2, 1\}, \alpha \neq 0, \text{(LI)}$
- $\mathcal{X}_{49}^4 = \text{Lin}\{F^{(2)}K^4 + \alpha F^{(1)}K^4 E^{(1)} + \alpha^2 K^4 E^{(2)} + \beta F^{(1)}K^4 + \alpha\beta K^4 E^{(1)} + \gamma K^4, F^{(1)}K^2 + (\beta - \alpha\delta)K^2, K^2 E^{(1)} + \delta K^2, 1\}, \text{(LI)} \alpha \neq (q - q^{-3})\gamma - (q - q^{-1})\beta^2$

- $\mathcal{X}_{50}^4 = \text{Lin}\{F^{(2)}K^4 + \alpha F^{(1)}K^4E^{(1)} + \alpha^2 K^4E^{(2)} + \beta F^{(1)}K^4 + \alpha\beta K^4E^{(1)} + \gamma K^4, F^{(1)}K^2 + \alpha K^2E^{(1)} + \beta K^2, f_\mu, 1\}, \mu \neq 1,$
- $\mathcal{X}_{51}^4 = \text{Lin}\{K^4E^{(2)} + \alpha K^4E^{(1)} + \beta K^4, F^{(1)}K^2 + \gamma K^2, K^2E^{(1)} + \alpha K^2, 1\},$   
(LI)  $(1 + q^{-2})\beta \neq \alpha^2$
- $\mathcal{X}_{52}^4 = \text{Lin}\{K^4E^{(2)} + \alpha K^4E^{(1)} + \beta K^4, K^2E^{(1)} + \alpha K^2, f_\mu, 1\}, \mu \neq 1,$
- $\mathcal{X}_{53}^4 = \text{Lin}\{F^{(1)}f_\mu + \alpha f_\mu E^{(1)} + \beta f_\mu, F^{(1)}K^2 + \gamma K^2E^{(1)} + \delta K^2, f_{\mu q^{-1}}, 1\},$   
 $\mu \neq q,$   
(LI)  $\gamma \neq \alpha, \mu^2 \neq q^2$
- $\mathcal{X}_{54}^4 = \text{Lin}\{F^{(1)}f_\mu + \alpha f_\mu E^{(1)} + \beta f_\mu, K^2E^{(1)} + \gamma K^2, f_{\mu q^{-1}}, 1\}, \mu \neq q,$   
(LI)  $\mu^2 \neq q^2$
- $\mathcal{X}_{55}^4 = \text{Lin}\{F^{(1)}f_\mu + \alpha f_\mu, f_\mu E^{(1)} + \beta f_\mu, f_{\mu q^{-1}}, 1\}, \mu \neq q,$   
(LI)  $\mu^2 \neq q^2$
- $\mathcal{X}_{56}^4 = \text{Lin}\{F^{(1)}f_\mu + \alpha f_\mu E^{(1)} + \beta f_\mu, f_{\mu q^{-1}}, f_\nu, 1\}, \mu \neq q, \mu \neq q\nu, \nu \neq 1,$
- $\mathcal{X}_{57}^4 = \text{Lin}\{F^{(1)}K^2 + \alpha K^2E^{(1)} + \beta K^2, f_\mu E^{(1)} + \gamma f_\mu, f_{\mu q^{-1}}, 1\}, \mu \neq q,$   
(LI)  $\mu^2 \neq q^2$
- $\mathcal{X}_{58}^4 = \text{Lin}\{F^{(1)}K^2 + \alpha K^2, K^2E^{(1)} + \beta K^2, f_\mu, 1\}, \mu \neq 1,$   
(LI)  $\mu^2 \neq 1$
- $\mathcal{X}_{59}^4 = \text{Lin}\{F^{(1)}K^2 + \alpha K^2E^{(1)} + \beta K^2, f_\mu, f_\nu, 1\}, \mu \neq \nu, \mu \neq 1, \nu \neq 1,$
- $\mathcal{X}_{60}^4 = \text{Lin}\{K^2E^{(1)} + \alpha K^2, f_\mu, f_\nu, 1\}, \mu \neq \nu, \mu \neq 1, \nu \neq 1,$
- $\mathcal{X}_{61}^4 = \text{Lin}\{f_\mu E^{(1)} + \alpha f_\mu, K^2E^{(1)} + \beta K^2, f_{\mu q^{-1}}, 1\}, \mu \neq q,$
- $\mathcal{X}_{62}^4 = \text{Lin}\{f_\mu E^{(1)} + \alpha f_\mu, f_{\mu q^{-1}}, f_\nu, 1\}, \mu \neq q, \mu \neq q\nu, \nu \neq 1,$
- $\mathcal{X}_{63}^4 = \text{Lin}\{f_{\mu_1}, f_{\mu_2}, f_{\mu_3}, 1\}, \mu_i \neq \mu_j, \mu_i \neq 1 \text{ for any } i \neq j.$

## 5 Further restrictions on the calculus

### 5.1 Generators of the FODC

Let  $\Gamma$  be a 3-dimensional left-covariant differential calculus on  $\mathrm{SL}_q(2)$ . We shall look for calculi  $\Gamma$  satisfying the following additional condition:

- (LI) The left-invariant one-forms  $\omega(u_2^1)$ ,  $\omega(u_1^2)$  and  $\omega(u_1^1 - u_2^2)$  generate  $\Gamma$  as a left  $\mathcal{A}$ -module.

It is simple to check that this condition is fulfilled if and only if the subspace  $\bar{\mathcal{X}}$  of the linear functionals on the matrix elements of the fundamental corepresentation  $u$  is four dimensional. Note that the classical differential calculus on  $\mathrm{SL}(2)$  obviously has this property. All coideals from Subsection 4.3 satisfying condition (LI) are marked with the label (LI). The necessary and sufficient conditions for the parameter values are indicated after the label.

### 5.2 Universal (higher order) differential calculi

In the previous section we have seen that there is a very large number of left-covariant first order differential calculi on the quantum group  $\mathrm{SL}_q(2)$ . Now the corresponding left-covariant universal (higher order) differential calculi will be considered, too. We require that

- the dimension of the space of left-invariant differential 2-forms is at least 3.

Recall that any differential calculus is a quotient of the universal differential calculus. Hence if there is a calculus such that the dimension of the space of its left-invariant differential 2-forms is equal to 3 then the universal calculus satisfies the above condition.

In order to study this condition we use Lemma 2 and the list in Subsection 4.3. As a sample let us consider  $\mathcal{X} = \mathcal{X}_8^4$ ,  $\gamma \neq 2\alpha$ . We then have

$$\begin{aligned}\bar{\mathcal{X}} &= \{X_1, X_2, X_3, X_4\} \\ &= \{FKG + \alpha KEG + \beta K^2G + \gamma KE + \delta K^2, G, FK + \alpha KE + \beta K^2, 1\}.\end{aligned}$$

For the elements  $m(X_i \otimes X_j)$  we obtain

$$X_1X_1 = q^{-1}F^2K^2G^2 + (1 + q^{-2})\alpha FK^2EG^2 + q^{-1}\alpha^2K^2E^2G^2 \quad (16)$$

+ terms of lower degree

$$X_1X_2 = FK^2G^2 + \alpha KEG^2 + \beta K^2G^2 + \gamma KEG + \delta K^2G \quad (17)$$

$$X_1X_3 = q^{-1}F^2K^2G + (1 + q^{-2})\alpha FK^2EG + q^{-1}\alpha^2K^2E^2G \quad (18)$$

+ terms of lower degree

$$X_1X_4 = FK^2G + \alpha KEG + \beta K^2G + \gamma KE + \delta K^2 \quad (19)$$

$$X_2X_1 - X_1X_2 + 2X_1X_4 = 4\alpha KEG + 2\beta K^2G + 4\gamma KE + 2\delta K^2 \quad (20)$$

$$X_2X_2 = G^2 \quad (21)$$

$$X_2X_3 - X_1X_4 + 2X_3X_4 = (-\gamma + 4\alpha)KE + (-\delta + 2\beta)K^2 \quad (22)$$

$$X_2X_4 = G \quad (23)$$

$$X_3X_1 - X_1X_3 - 2X_3^2 = ((1 - q^{-2})\gamma - 4\alpha)FK^2E - 4q^{-1}\alpha^2K^2E^2 \quad (24)$$

+ terms of lower degree

$$X_3X_2 - X_1X_4 = -\gamma KE - \delta K^2 \quad (25)$$

$$X_3X_3 = q^{-1}F^2K^2 + (1 + q^{-2})\alpha FK^2E + q^{-1}\alpha^2K^2E^2 + (1 + q^{-2})\beta FK^3$$

+  $(1 + q^{-2})\alpha\beta K^3E + (\beta^2 + \alpha/(q - q^{-1}))K^4 - \alpha/(q - q^{-1})$  (26)

$$X_3X_4 = FK + \alpha KE + \beta K^2 \quad (27)$$

$$X_4X_1 - X_1X_4 = 0 \quad (28)$$

$$X_4X_2 - X_2X_4 = 0 \quad (29)$$

$$X_4X_3 - X_3X_4 = 0 \quad (30)$$

$$X_4X_4 = 1 \quad (31)$$

From Lemma 2 we conclude that there are at least  $3 + \dim \mathcal{X} = 6$  linearly independent relations in  $\bar{\mathcal{X}}^2$ . Hence there are at most 10 linearly independent elements in  $\bar{\mathcal{X}}^2$ . Because of the PBW-theorem the 9 elements in (16)–(19), (21), (23), (26), (27) and (31) are linearly independent. Suppose that  $\alpha$  is nonzero. Then (20) and the difference (22)–(25) are further linearly independent elements which is a contradiction. If  $\alpha = 0$  then  $\gamma \neq 0$  because of (LI).

Therefore, (22) and (24) are two additional linearly independent elements in  $\bar{\mathcal{X}}^2$  and we obtain again a contradiction.

The same procedure can be applied to all right coideals of the list in Subsection 4.3. In order to carry out these computations we used the computer algebra program FELIX [1]. The result is the following (the number of the general coideal and the corresponding parameter values are given in parenthesis):

1. (47;  $\alpha = \beta = \gamma = 0, \delta = q^5/(q^2 - 1)^2$ )  
 $\bar{\mathcal{X}} = \{FK^2E + q^5/(q^2 - 1)^2K^4, FK, KE, 1\}$
2. (53;  $\alpha = 0, \beta \neq 0, \gamma = -(q - q^{-1})^2\beta^2, \delta = (1 + q)\beta, \mu = q^{1/2}$ )  
 $\bar{\mathcal{X}} = \{F + \beta K, FK - (q - q^{-1})^2\beta^2KE + (1 + q)\beta K^2, K^{-1}, 1\}$
3. (53;  $\alpha = 0, \beta \neq 0, \gamma = -(q - q^{-1})^2\beta^2, \delta = (1 + q)\beta, \mu = -q^{1/2}$ )  
 $\bar{\mathcal{X}} = \{F\varepsilon_- + \beta\varepsilon_-K, FK - (q - q^{-1})^2\beta^2KE + (1 + q)\beta K^2, \varepsilon_-K^{-1}, 1\}$
4. (53;  $\alpha = 0, \beta \neq 0, \gamma = (q - q^{-1})^2\beta^2, \delta = (1 - q)\beta, \mu = iq^{1/2}$ )  
 $\bar{\mathcal{X}} = \{Ff_i + \beta f_iK, FK + (q - q^{-1})^2\beta^2KE + (1 - q)\beta K^2, f_iK^{-1}, 1\}$
5. (53;  $\alpha = 0, \beta \neq 0, \gamma = (q - q^{-1})^2\beta^2, \delta = (1 - q)\beta, \mu = -iq^{1/2}$ )  
 $\bar{\mathcal{X}} = \{Ff_{-i} + \beta f_{-i}K, FK + (q - q^{-1})^2\beta^2KE + (1 - q)\beta K^2, f_{-i}K^{-1}, 1\}$
6. (53;  $\alpha \neq 0, \gamma = -\alpha, \beta = \delta = 0, \mu = iq$ )  $\dim_{\mathbf{u}} \Gamma^{\wedge 2} = 4$   
 $\bar{\mathcal{X}} = \{Ff_iK + \alpha f_iKE, FK - \alpha KE, f_i, 1\}$
7. (53;  $\alpha \neq 0, \gamma = -\alpha, \beta = \delta = 0, \mu = -iq$ )  $\dim_{\mathbf{u}} \Gamma^{\wedge 2} = 4$   
 $\bar{\mathcal{X}} = \{Ff_{-i}K + \alpha f_{-i}KE, FK - \alpha KE, f_{-i}, 1\}$
8. (54;  $\alpha = \beta = \gamma = 0, \mu = q^3$ )  
 $\bar{\mathcal{X}} = \{FK^5, KE, K^4, 1\}$
9. (54;  $\alpha = \beta = \gamma = 0, \mu = q^{-1}$ )  
 $\bar{\mathcal{X}} = \{FK^{-3}, KE, K^{-4}, 1\}$

10. (55;  $\alpha = \beta = 0, \mu = 1$ )  
 $\bar{\mathcal{X}} = \{FK^{-1}, K^{-1}E, K^{-2}, 1\}$
11. (55;  $\alpha = \beta = 0, \mu = -1$ )  
 $\bar{\mathcal{X}} = \{F\varepsilon_-K^{-1}, \varepsilon_-K^{-1}E, \varepsilon_-K^{-2}, 1\}$
12. (55;  $\alpha = \beta = 0, \mu = q^{-1}$ )  
 $\bar{\mathcal{X}} = \{FK^{-3}, K^{-3}E, K^{-4}, 1\}$
13. (57;  $\alpha = \beta = \gamma = 0, \mu = q^3$ )  
 $\bar{\mathcal{X}} = \{FK, K^5E, K^4, 1\}$
14. (57;  $\alpha = \beta = \gamma = 0, \mu = q^{-1}$ )  
 $\bar{\mathcal{X}} = \{FK, K^{-3}E, K^{-4}, 1\}$
15. (57;  $\gamma \neq 0, \alpha = -1/((q - q^{-1})^2\gamma^2), \beta = 1/((q - 1)(q^{-2} - 1)\gamma), \mu = q^{1/2}$ )  
 $\bar{\mathcal{X}} = \{FK + \alpha KE + \beta K^2, E + \gamma K, K^{-1}, 1\}$
16. (57;  $\gamma \neq 0, \alpha = -1/((q - q^{-1})^2\gamma^2), \beta = 1/((q - 1)(q^{-2} - 1)\gamma), \mu = -q^{1/2}$ )  
 $\bar{\mathcal{X}} = \{FK + \alpha KE + \beta K^2, \varepsilon_-E + \gamma\varepsilon_-K, \varepsilon_-K^{-1}, 1\}$
17. (58;  $\alpha = \beta = 0, \mu = q^2$ )  
 $\bar{\mathcal{X}} = \{FK, KE, K^4, 1\}$
18. (58;  $\alpha = \beta = 0, \mu = q$ )  
 $\bar{\mathcal{X}} = \{FK, KE, K^2, 1\}$
19. (58;  $\alpha \neq 0, \beta = -q^3/((q^2 - 1)^2\alpha), \mu = q^{-1}$ )  
 $\bar{\mathcal{X}} = \{FK + \alpha K^2, KE - q^3/((q^2 - 1)^2\alpha)K^2, K^{-2}, 1\}$
20. (58;  $\alpha = \beta = 0, \mu = -q$ )  
 $\bar{\mathcal{X}} = \{FK, KE, \varepsilon_-K^2, 1\}$

**Remark.** The solutions 6 and 7 have the property  $\dim_{\mathfrak{u}} \Gamma^{\wedge 2} = 4$ . In all other cases we have  $\dim_{\mathfrak{u}} \Gamma^{\wedge 2} = 3$ . ■

### 5.3 Hopf algebra automorphisms

Let  $\mathcal{A}$  be a Hopf algebra and  $\varphi$  an automorphism of  $\mathcal{A}$ . Let  $\Gamma$  be an  $\mathcal{A}$ -bimodule. Then the actions  $\cdot : \mathcal{A} \times \Gamma \rightarrow \Gamma$ ,  $a \cdot \rho := \varphi(a)\rho$  and  $\cdot : \Gamma \times \mathcal{A} \rightarrow \Gamma$ ,  $\rho \cdot a := \rho\varphi(a)$  determine another  $\mathcal{A}$ -bimodule structure on  $\Gamma$ , since  $\varphi$  is an algebra homomorphism. Let  $\Gamma$  be a left  $\mathcal{A}$ -module. Then the mapping  $\Delta'_L : \Gamma \rightarrow \mathcal{A} \otimes \Gamma$ ,  $\Delta'_L(\rho) = \varphi(\rho_{(-1)}) \otimes \rho_{(0)}$  determines a second left  $\mathcal{A}$ -comodule structure on  $\Gamma$ , since  $\varphi$  is a comodule homomorphism.

If  $(\Gamma, d)$  is a left-covariant FODC over  $\mathcal{A}$ , set  $\Gamma_\varphi := \Gamma$  with module action  $\cdot$  and comodule mapping  $\Delta'_L$ . Further define  $d_\varphi : \mathcal{A} \rightarrow \Gamma$ ,  $d_\varphi a := d\varphi(a)$ . Then  $(\Gamma_\varphi, d_\varphi)$  is a left-covariant FODC over  $\mathcal{A}$ . Indeed,  $d_\varphi$  satisfies the Leibniz rule and

$$\text{Lin}\{a \cdot d_\varphi b \mid a, b \in \mathcal{A}\} = \text{Lin}\{\varphi(a)d\varphi(b) \mid a, b \in \mathcal{A}\} = \text{Lin}\{adb \mid a, b \in \mathcal{A}\}$$

since  $\varphi$  is invertible.

**Definition 1.** Let  $(\Gamma, d)$  be a left-covariant FODC over the Hopf algebra  $\mathcal{A}$ . We call  $(\Gamma, d)$  *Hopf-invariant* if  $(\Gamma_\varphi, d_\varphi)$  is isomorphic to  $(\Gamma, d)$  for all Hopf algebra automorphism  $\varphi$  of  $\mathcal{A}$ .

**Proposition 5.** Let  $(\Gamma, d)$  be a left-covariant FODC over  $\mathcal{A}$ . The following statements are equivalent:

- (i)  $(\Gamma, d)$  is Hopf-invariant.
- (ii)  $\varphi(\mathcal{R}_\Gamma) = \mathcal{R}_\Gamma$  for any Hopf algebra automorphism  $\varphi$  of  $\mathcal{A}$ .
- (iii)  $\varphi(\mathcal{X}_\Gamma) = \mathcal{X}_\Gamma$  for any Hopf algebra automorphism  $\varphi$  of  $\mathcal{A}^\circ$  which is implemented by  $\mathcal{A}$ .

**Proof.** It suffices to show that for any Hopf algebra automorphism  $\varphi$  of  $\mathcal{A}$  we have  $\mathcal{R}_{\Gamma_\varphi} = \varphi^{-1}(\mathcal{R}_\Gamma)$  and  $\mathcal{X}_{\Gamma_\varphi} = \varphi(\mathcal{X}_\Gamma)$ . For this we compute

$$\omega_\varphi(a) := S(a_{(1)}) \cdot d_\varphi a_{(2)} = \varphi(S(a_{(1)}))d\varphi(a_{(2)}) = S(\varphi(a)_{(1)})d\varphi(a)_{(2)} = \omega(\varphi(a))$$

for any  $a \in \mathcal{A}$ . Hence (with  $\mathcal{A}^+ = \mathcal{A} \cap \ker \varepsilon$ )

$$\mathcal{R}_{\Gamma_\varphi} = \{a \in \mathcal{A}^+ \mid \omega_\varphi(a) = 0\}$$

$$= \{a \in \mathcal{A}^+ \mid \omega(\varphi(a)) = 0\} = \{\varphi^{-1}(b) \in \mathcal{A}^+ \mid \omega(b) = 0\} = \varphi^{-1}(\mathcal{R}_\Gamma)$$

and (with  $\mathcal{A}^{\circ+} := \mathcal{A}^\circ \cap \ker \varepsilon$ )

$$\begin{aligned} \mathcal{X}_{\Gamma_\varphi} &= \{X \in \mathcal{A}^{\circ+} \mid X(\mathcal{R}_{\Gamma_\varphi}) = 0\} = \{X \in \mathcal{A}^{\circ+} \mid X(\varphi^{-1}(\mathcal{R}_\Gamma)) = 0\} \\ &= \{Y \circ \varphi \in \mathcal{A}^{\circ+} \mid Y(\mathcal{R}_\Gamma) = 0\} = \varphi(\mathcal{X}_\Gamma). \end{aligned}$$

■

It is easy to prove that any Hopf algebra automorphism  $\varphi$  of  $\mathcal{O}(\mathrm{SL}_q(2))$  is given by  $\varphi(u_1^1) = u_1^1$ ,  $\varphi(u_2^2) = u_2^2$ ,  $\varphi(u_2^1) = \alpha u_2^1$ ,  $\varphi(u_1^2) = \alpha^{-1} u_1^2$ , where  $\alpha \in \mathbb{C}^\times$  (see also [7, Section 4.1.2]). For this  $\varphi =: \varphi_\alpha$  we obtain from (13) the formulas

$$\varphi_\alpha(E) = \alpha^{-1}E, \quad \varphi_\alpha(F) = \alpha F, \quad \varphi_\alpha(f_\mu) = f_\mu, \quad \mu \in \mathbb{C}^\times. \quad (32)$$

**Proposition 6.** *A left-covariant FODC  $\Gamma$  over  $\mathcal{O}(\mathrm{SL}_q(2))$  is Hopf-invariant if and only if its quantum tangent space  $\mathcal{X}_\Gamma$  is generated by elements which are homogeneous with respect to the  $\mathbb{Z}$ -grading of  $\mathcal{U}$ .*

**Proof.** Clearly  $\varphi_\alpha(X) = \alpha^{-n}X$  for any  $X \in \mathcal{X}$ ,  $\deg X = n$ . Therefore the condition of the Proposition is sufficient.

Conversely, suppose that  $X = \sum_{i=1}^k \lambda_i X_i \in \mathcal{X}$ ,  $k > 1$ , where  $\lambda_i \in \mathbb{C}^\times$ ,  $X_i \in \mathcal{X}$ ,  $\deg X_i = n_i$  and  $n_i \neq n_j$  for any  $i \neq j$ . If  $(\Gamma, d)$  is Hopf-invariant then we have  $\mathcal{X} \ni \varphi_q(X) - q^{-n_k}X = \sum_{i=1}^{k-1} (q^{-n_i} - q^{-n_k})X_i$ . Since  $q$  is not a root of unity, we obtain that  $\sum_{i=1}^{k-1} \mu_i X_i \in \mathcal{X}$ ,  $\mu_i \in \mathbb{C}^\times$ . By induction on  $k$  one easily proves that  $X_i \in \mathcal{X}$  for any  $i$ . ■

**Corollary 7.** *The items 1, 8-14, 17, 18 and 20 are precisely the quantum tangent spaces of the list in Subsection 5.2 which are Hopf-invariant.*

## 6 More structures on left-covariant differential calculi on $\mathrm{SL}_q(2)$

The Hopf algebra  $\mathcal{O}(\mathrm{SL}_q(2))$  admits 3 non-equivalent real forms. Namely,

- $q \in \mathbb{R}; (u_1^1)^* = u_2^2, (u_2^1)^* = -qu_1^2, (u_1^2)^* = -q^{-1}u_2^1, (u_2^2)^* = u_1^1;$
- $q \in \mathbb{R}; (u_1^1)^* = u_2^2, (u_2^1)^* = qu_1^2, (u_1^2)^* = q^{-1}u_2^1, (u_2^2)^* = u_1^1;$
- $|q| = 1; (u_j^i)^* = u_j^i$  for any  $i, j = 1, 2.$

The Hopf  $*$ -algebras corresponding to them are  $\mathcal{O}(\mathrm{SU}_q(2)), \mathcal{O}(\mathrm{SU}_q(1, 1))$  and  $\mathcal{O}(\mathrm{SL}_q(2, \mathbb{R}))$ . Let us introduce the dual involution on  $\mathcal{U}$  in the way  $f^*(a) := \overline{f(S(a)^*)}$ . Then the corresponding  $*$ -structures on  $\mathcal{U}$  are given by

- $E^* = F, F^* = E, G^* = G, f_\mu^* = f_{\bar{\mu}},$
- $E^* = -F, F^* = -E, G^* = G, f_\mu^* = f_{\bar{\mu}},$
- $E^* = -qE, F^* = -q^{-1}F, G^* = -G, f_\mu^* = f_{\bar{\mu}^{-1}},$

respectively. A FODC  $(\Gamma, d)$  over a Hopf  $*$ -algebra  $\mathcal{A}$  is called a  $*$ -calculus if there exists an involution  $*$  :  $\Gamma \rightarrow \Gamma$  such that  $(a(db)c)^* = c^*(db^*)a^*$  for any  $a, b, c \in \mathcal{A}$ .

**Proposition 8.** [7] *Let  $(\Gamma, d)$  be a finite-dimensional left-covariant FODC over a Hopf  $*$ -algebra  $\mathcal{A}$ . Then  $(\Gamma, d)$  is a  $*$ -calculus if and only if its quantum tangent space  $\mathcal{X}_\Gamma$  is  $*$ -invariant.*

Let  $\Gamma$  be a left-covariant bimodule over a Hopf algebra  $\mathcal{A}$ . An invertible linear mapping  $\sigma : \Gamma \otimes_{\mathcal{A}} \Gamma \rightarrow \Gamma \otimes_{\mathcal{A}} \Gamma$  is called a *braiding of  $\Gamma$*  if  $\sigma$  is a homomorphism of  $\mathcal{A}$ -bimodules, commutes with the left coaction on  $\Gamma$  and satisfies the braid relation

$$(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id}) = (\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)$$

on  $\Gamma^{\otimes 3}$ . If  $(\Gamma, d)$  is a left-covariant differential calculus over  $\mathcal{A}$  then we require that  $(\mathrm{id} - \sigma)(\mathcal{S} \cap \Gamma^{\otimes 2}) = 0$ . Such a braiding neither needs to exist nor it is unique for a given left-covariant differential calculus over  $\mathcal{A}$ .

Let  $(\Gamma, d)$  be a left-covariant FODC over  $\mathcal{A}$ . Fix a basis  $\{X_i \mid i = 1, \dots, n\}$  of its quantum tangent space  $\mathcal{X}_\Gamma$  and let  $\{\omega_i \mid i = 1, \dots, n\}$  be the dual basis

of  $\Gamma_L$ . If  $\sigma$  is a braiding of  $\Gamma$  with  $\sigma(\omega_i \otimes \omega_j) = \sigma_{ij}^{kl} \omega_k \otimes \omega_l$  then we can define a bilinear mapping  $[\cdot, \cdot] : \mathcal{X}_\Gamma \times \mathcal{X}_\Gamma \rightarrow \mathcal{X}_\Gamma$  by

$$[X_i, X_j] := X_i X_j - \sigma_{kl}^{ij} X_k X_l. \quad (33)$$

This mapping can be viewed as a generalization of the Lie bracket of the left-invariant vector fields if it also satisfies a generalized Jacobi identity. For the calculi and braidings described below the mapping  $\beta := [\cdot, \cdot]$  fulfills the equation

$$(\beta \circ (\beta \otimes \text{id}) - \beta \circ (\text{id} \otimes \beta))A_3^t = 0 \quad (34)$$

where  $A_3^t(X_i \otimes X_j \otimes X_k) = (\text{id} - \sigma_{12})(\text{id} - \sigma_{23} + \sigma_{23}\sigma_{12})_{rst}^{ijk} X_r \otimes X_s \otimes X_t$ . Unfortunately, such an equation does not hold for bicovariant differential calculi in general.

## 7 The calculi in detail

The main result of the considerations of the preceding sections is the following.

**Theorem 9.** *Suppose that  $q$  is a transcendental complex number. Let  $(\Gamma, d)$  be a left-covariant FODC over  $\mathcal{O}(\text{SL}_q(2))$  having the following properties:*

- $\Gamma$  is a 3-dimensional left-covariant bimodule
- the left-invariant one-forms  $\omega(u_2^1), \omega(u_1^2)$  and  $\omega(u_1^1 - u_2^2)$  form a basis of the left module  $\Gamma$
- the universal differential calculus  ${}_u\Gamma^\wedge$  associated to  $\Gamma$  satisfies the inequality  $\dim {}_u\Gamma^{\wedge 2} \geq 3$
- $\Gamma$  is Hopf-invariant.

Then  $\Gamma$  is isomorphic to one of the 11 left-covariant FODC 1,8-14,17,18,20 of the list in Subsection 5.2.

**Remarks.** 1. The calculus 17 is isomorphic to the 3D-calculus of Woronowicz [13]. Two other 3-dimensional calculi appear in [9]. They are isomorphic to the calculus 13 (for  $r = 2$ ) and 8 (for  $r = 3$ ), respectively. The calculi 10 and 11 are subcalculi of the bicovariant  $4D_+$ - and  $4D_-$ -calculus, respectively (see also [7, Section 14.2.4]). One can easily check all these isomorphisms by comparing the corresponding right ideals  $\mathcal{R}_\Gamma$ .

2. In [8, Section 7] the notion of *elementary* left-covariant FODC was introduced. It is an easy computation to show that the six calculi 10–12,17,18 and 20 in Theorem 9 are elementary, while the others are not. ■

After the last section we list some important facts about these 11 calculi. First we fix the basis  $\omega_H := \omega((u_1^1 - u_2^2)/2)$ ,  $\omega_X := \omega(u_2^1)$ ,  $\omega_Y := \omega(u_1^2)$  in  $\Gamma_L$  and determine the dual basis  $\{H, X, Y\}$  (i.e.  $\omega(a) = H(a)\omega_H + X(a)\omega_X + Y(a)\omega_Y$  for any  $a \in \mathcal{O}(\mathrm{SL}_q(2))$ ). We examine whether or not the calculus is a  $*$ -calculus with respect to the three given involutions on  $\mathcal{O}(\mathrm{SL}_q(2))$ . We compute the pairing between the quantum tangent space and linear (fundamental representation) and quadratic elements of  $\mathcal{O}(\mathrm{SL}_q(2))$ . Using this the generators of the right ideal  $\mathcal{R}_\Gamma$  are computed.

The quadratic-linear relations between generators of  $\mathcal{X}_\Gamma$  are given. Because of their simple form one can easily show that the algebra without unit generated by the elements  $H, X$ , and  $Y$  and these relations has the PBW-basis  $\{H^{n_1}X^{n_2}Y^{n_3} \mid n_1, n_2, n_3 \in \mathbb{N}_0, n_1 + n_2 + n_3 > 0\}$ . This proves also that  $\dim_{\mathfrak{u}} \Gamma^{\wedge 3} = 1$  and  $\dim_{\mathfrak{u}} \Gamma^{\wedge k} = 0$  for any  $k > 3$  and for each of the 11 calculi.

The matrix  $(f_j^i)$  describes the commutation rules between one-forms and functions (see equation (4)), where  $i, j \in \{H, X, Y\}$ . Finally, the generators of the right ideal and equation (7) determine the vector space of left-invariant symmetric 2-forms. If there exists a braiding  $\sigma$  of the left-covariant bimodule  $\Gamma$  such that  $(\mathrm{id} - \sigma)(\mathcal{S} \cap \Gamma_L^{\otimes 2}) = 0$  then we give one of them by its eigenvalues and eigenspaces. It was determined by means of the computer algebra

program FELIX [1].

## 8 Cohomology

Let us fix one of the differential calculi of Theorem 9. The differential mapping  $d : \Gamma^\wedge \rightarrow \Gamma^\wedge$  satisfies the equation  $d^2 = 0$ . Hence it defines a complex

$$\{0\} \xrightarrow{d} \mathcal{A} \xrightarrow{d} \Gamma \xrightarrow{d} \Gamma^{\wedge 2} \xrightarrow{d} \Gamma^{\wedge 3} \xrightarrow{d} \Gamma^{\wedge 4} = \{0\}. \quad (35)$$

Now the corresponding cohomology spaces will be determined. In [13] Woronowicz introduced a method to do this. Let us recall his strategy.

The first observation is that the algebra  $\mathcal{A} = \mathcal{O}(\mathrm{SL}_q(2))$  has a vector space basis

$$\{v_{ij}^\lambda \mid \lambda \in \frac{1}{2}\mathbb{N}_0; -\lambda \leq i, j \leq \lambda; \lambda - i, \lambda - j \in \mathbb{N}_0\} \quad (36)$$

such that  $\Delta(v_{ij}^\lambda) = v_{ik}^\lambda \otimes v_{kj}^\lambda$  for any  $\lambda, i, j$ . Let us fix such a basis. Then for  $\rho_j \in \Gamma_L^\wedge$  we obtain

$$d(v_{ij}^\lambda \rho_j) = v_{ik}^\lambda X_r(v_{kj}^\lambda) \omega_r \wedge \rho_j + v_{ij}^\lambda d\rho_j. \quad (37)$$

The elements  $\sum_{r,j} X_r(v_{kj}^\lambda) \omega_r \wedge \rho_j + d\rho_k$  are left-invariant. Hence the differential mapping  $d$  preserves the direct sum decomposition

$$\Gamma^\wedge = \bigoplus_{\lambda, i} C(v_i^\lambda) \Gamma_L^\wedge, \quad (38)$$

where  $C(v_i^\lambda) = \mathrm{Lin}\{v_{kl}^\mu \mid \mu = \lambda, k = i\}$ . Moreover, formula (37) for the differential on  $C(v_i^\lambda) \Gamma_L^\wedge$  does not depend on  $i$ . Therefore, for any  $\lambda \in \frac{1}{2}\mathbb{N}_0$  we obtain  $2\lambda$  isomorphic differential complexes.

Let  $V^\lambda = \mathrm{Lin}\{e_\mu^\lambda \mid -\lambda \leq \mu \leq \lambda; \lambda - \mu \in \mathbb{N}_0\}$  be the representation of  $\mathcal{A}^\circ$  defined by  $f.e_\mu^\lambda = e_\nu^\lambda f(v_{\nu\mu}^\lambda)$  for any  $f \in \mathcal{A}^\circ$ . It is isomorphic to the representation given by

$$E.e_\nu^\lambda = [\lambda + \nu]e_{\nu-1}^\lambda, \quad F.e_\nu^\lambda = [\lambda - \nu]e_{\nu+1}^\lambda,$$

$$f_\mu \cdot e_\nu^\lambda = \mu^{-2\nu} e_\nu^\lambda, \quad G \cdot e_\nu^\lambda = -2\nu e_\nu^\lambda. \quad (39)$$

Because of the left-covariance of the differential calculus,  $V^\lambda$  induces a representation of the quantum tangent space  $\mathcal{X}_r$  of the differential calculus. We obtain a new complex

$$\begin{aligned} \{0\} \xrightarrow{d} V^\lambda \xrightarrow{d} V^\lambda \otimes \Gamma_L \xrightarrow{d} V^\lambda \otimes \Gamma_L^{\wedge 2} \\ \xrightarrow{d} V^\lambda \otimes \Gamma_L^{\wedge 3} \xrightarrow{d} \{0\}, \end{aligned} \quad (40)$$

where

$$d(e_\mu^\lambda \otimes \rho_\mu) = e_\nu^\lambda X_r(v_{\nu\mu}^\lambda) \otimes \omega_r \wedge \rho_\mu + e_\mu^\lambda \otimes d\rho_\mu. \quad (41)$$

**Lemma 10.** *The cohomology spaces of the complex (41) are isomorphic to  $H_\lambda^0 = H_\lambda^3 = \mathbb{C}^p$ ,  $H_\lambda^1 = H_\lambda^2 = \{0\}$ , where  $p = 1$  for  $\lambda = 0$  and  $p = 0$  otherwise.*

**Theorem 11.** *Let  $\Gamma$  be one of the differential calculi of Theorem 9. Then the cohomology spaces of the differential complex (35) are isomorphic to*

$$H^0 = H^3 = \mathbb{C}^p, \quad H^1 = H^2 = \{0\}, \quad (42)$$

where  $p = 1$  for  $\lambda = 0$  and  $p = 0$  otherwise.

**Proof.** The assertion follows from Lemma 10 and the preceding considerations. ■

**Proof of the Lemma.** For  $\lambda = 0$  we have  $de_0^0 = 0$  and  $d(e_0^0 \otimes \omega) \neq 0$  for any  $\omega \in \Gamma_L$ . Since  $\dim \Gamma_L^{\wedge 2} = \dim \Gamma_L (= 3) < \infty$ , the mapping  $d : \Gamma_L \rightarrow \Gamma_L^{\wedge 2}$  is also surjective. Therefore, for any  $\xi \in \Gamma_L^{\wedge 2}$  we obtain  $d\xi (= d^2\omega) = 0$ . Hence the assertion of the lemma for  $\lambda = 0$  is valid.

Let now  $\lambda \neq 0$ . We set

$$\partial(e_\mu^\lambda \otimes \bigwedge_{j=1}^k \omega_{i_j}) := \mu + \sum_{j=1}^k \partial(\omega_{i_j}), \quad \text{where } \partial(\omega_H) = 0, \partial(\omega_Y) = -\partial(\omega_X) = 1. \quad (43)$$

Then  $\partial$  defines a direct sum decomposition of  $V^\lambda \otimes \Gamma_L^\wedge$  and  $d$  preserves this decomposition:  $\partial(d\rho) = \partial(\rho)$  for any homogeneous element  $\rho$  of  $V^\lambda \otimes \Gamma_L^\wedge$ . Hence it suffices to check that the sequence (40) restricted to homogeneous elements of the same degree  $\mu$  is exact for any  $\lambda \in \frac{1}{2}\mathbb{N}$  and any  $\mu$  with  $\lambda - \mu \in \mathbb{Z}$ .

For  $\rho = e_\mu^\lambda$  the equation  $d\rho(= X_i.e_\mu^\lambda \otimes \omega_i) = 0$  implies that  $X.e_\mu^\lambda = Y.e_\mu^\lambda = 0$ . Hence  $\mu = -\lambda$  and  $\mu = \lambda$ . This is a contradiction to  $\lambda \neq 0$ , so we have  $H_\lambda^0 = \{0\}$ .

Since  $d\rho = 0$  for any  $\rho \in V^\lambda \otimes \Gamma_L^{\wedge 3}$  we have to show that  $\rho = d\xi$  for some  $\xi \in V^\lambda \otimes \Gamma_L^{\wedge 2}$ . If  $\partial(\rho) = \mu$  then  $-\lambda \leq \mu \leq \lambda$  and  $\rho$  is a constant multiple of  $e_\mu^\lambda \otimes \omega_H \wedge \omega_X \wedge \omega_Y$ . One can take  $\xi_1 := c^{-1}e_{\mu+1}^\lambda \otimes \omega_H \wedge \omega_X$  or  $\xi_2 := c^{-1}e_{\mu-1}^\lambda \otimes \omega_H \wedge \omega_Y$  with some  $c \in \mathbb{C}^\times$ . Indeed, because of  $\omega_H \wedge \omega_H \wedge \omega_X = \omega_X \wedge \omega_H \wedge \omega_X = d(\omega_H \wedge \omega_X) = 0$  the element  $d(e_{\mu+1}^\lambda \otimes \omega_H \wedge \omega_X) = Y.e_{\mu+1}^\lambda \otimes \omega_Y \wedge \omega_H \wedge \omega_X$  is a nonzero multiple of  $\rho$  for  $\mu \neq \lambda$ . Similarly,  $d(c\xi_2)$  is a nonzero multiple of  $\rho$  for  $\mu \neq -\lambda$ . Therefore,  $H_\lambda^3 = \{0\}$ .

Now we prove that  $H_\lambda^1 = \{0\}$ . It is easy to see that if  $\partial\omega = \mu$  for a nonzero  $\omega \in V^\lambda \otimes \Gamma_L$  then  $|\mu| \leq \lambda+1$ . Hence we have to show that  $d\omega \neq 0$  if  $|\mu| = \lambda+1$  and that the vector space  $\{\omega \in V^\lambda \otimes \Gamma_L \mid \partial\omega = \mu, d\omega = 0\}$  is one-dimensional for  $|\mu| \leq \lambda$ . We define  $V_1^{\lambda\mu} := \{\omega \in V^\lambda \otimes \Gamma_L \mid \partial\omega = \mu\}$ . Since  $e_\lambda^\lambda \otimes \omega_Y$  generates the vector space  $V_1^{\lambda, \lambda+1}$  we have to show that  $H.e_\lambda^\lambda \otimes \omega_H \wedge \omega_Y + e_\lambda^\lambda \otimes d\omega_Y \neq 0$ . Similarly,  $d(e_{-\lambda}^\lambda \otimes \omega_X) = H.e_{-\lambda}^\lambda \otimes \omega_H \wedge \omega_X + e_{-\lambda}^\lambda \otimes d\omega_X$  must be nonzero. Using the explicit formulas for the quantum tangent space, for the differentials  $d\omega_X$  and  $d\omega_Y$  and (39) this is easily done.

Secondly, the elements  $e_\mu^\lambda \otimes \omega_H$ ,  $e_{\mu+1}^\lambda \otimes \omega_X$  and  $e_{\mu-1}^\lambda \otimes \omega_Y$  generate the vector space  $V_1^{\lambda\mu}$ . We have  $\dim V_1^{\lambda\mu} = 2$  for  $|\mu| = \lambda$  and  $\dim V_1^{\lambda\mu} = 3$  for  $|\mu| < \lambda$ . Moreover,

$$\begin{aligned} d(e_\mu^\lambda \otimes \omega_H) &= X.e_\mu^\lambda \otimes \omega_X \wedge \omega_H + Y.e_\mu^\lambda \otimes \omega_Y \wedge \omega_H \\ &\quad + (H.e_\mu^\lambda \otimes \omega_H \wedge \omega_H + e_\mu^\lambda \otimes d\omega_H) \end{aligned} \quad (44)$$

$$d(e_{\mu+1}^\lambda \otimes \omega_X) = Y.e_{\mu+1}^\lambda \otimes \omega_Y \wedge \omega_X + (H.e_{\mu+1}^\lambda \otimes \omega_H \wedge \omega_X + e_{\mu+1}^\lambda \otimes d\omega_X) \quad (45)$$

$$d(e_{\mu-1}^\lambda \otimes \omega_Y) = X.e_{\mu-1}^\lambda \otimes \omega_X \wedge \omega_Y + (H.e_{\mu-1}^\lambda \otimes \omega_H \wedge \omega_Y + e_{\mu-1}^\lambda \otimes d\omega_Y) \quad (46)$$

Observe that  $\xi = c_{HX}(\xi) \otimes \omega_H \wedge \omega_X + c_{HY}(\xi) \otimes \omega_H \wedge \omega_Y + c_{XY}(\xi) \otimes \omega_X \wedge \omega_Y$  for any  $\xi \in V^\lambda \otimes \Gamma_L^{\wedge 2}$ , where  $c_{HX}(\xi), c_{HY}(\xi), c_{XY}(\xi) \in V^\lambda$ . Now if  $\mu = \lambda$  then by (46)  $c_{XY}(d(e_{\lambda-1}^\lambda \otimes \omega_Y)) = X.e_{\lambda-1}^\lambda \neq 0$  and therefore the range of  $d$  is at least one-dimensional. Similarly, by (45)  $c_{XY}(d(e_{\mu+1}^\lambda \otimes \omega_X)) \neq 0$  for  $\mu = -\lambda$ . Hence  $\dim \ker d \upharpoonright V_1^{\lambda\mu} \leq 1$  for  $|\mu| = \lambda$ . If  $|\mu| < \lambda$  then  $c_{HY}(d(e_\mu^\lambda \otimes \omega_H)) \neq 0$ ,  $c_{HY}(d(e_{\mu+1}^\lambda \otimes \omega_X)) = 0$  and  $c_{XY}(d(e_{\mu+1}^\lambda \otimes \omega_X)) \neq 0$ . Therefore,  $\dim d(V_1^{\lambda\mu}) \geq 2$  and so  $\dim \ker d \upharpoonright V_1^{\lambda\mu} \leq 1$ . Together,  $\dim \ker d \upharpoonright V_1^{\lambda\mu} \leq 1$  for  $|\mu| \leq \lambda$  and  $\ker d \upharpoonright V_1^{\lambda\mu} = \{0\}$  for  $|\mu| = \lambda + 1$ . Because of  $de_\mu^\lambda \neq 0$ ,  $\partial(de_\mu^\lambda) = \mu$  and  $d(de_\mu^\lambda) = 0$  we also have  $\dim \ker d \upharpoonright V_1^{\lambda\mu} \geq 1$  for  $|\mu| \leq \lambda$ . This means that  $H_\lambda^1 = \{0\}$ .

The last assertion,  $H_\lambda^2 = \{0\}$ , follows from dimension computations. ■

**1. Quantum tangent space  $\mathcal{X}_F$ :**

$$H := \frac{2(q^{-1} - q)}{q^4 + 1} \left( FK^2E + \frac{q^5(K^4 - 1)}{(q^2 - 1)^2} \right), \quad X := q^{-1/2}FK, \quad Y := q^{-1/2}KE$$

**Real forms:**  $\mathcal{O}(\mathrm{SU}_q(2)), \mathcal{O}(\mathrm{SU}_q(1, 1)), \mathcal{O}(\mathrm{SL}_q(2, \mathbb{R}))$

**Fund. repr.:**  $H = \frac{2}{q^2 + q^{-2}} \begin{pmatrix} q^{-2} & 0 \\ 0 & -q^2 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

**Relations:**

$$\begin{aligned} qHX - q^{-1}XH &= \frac{2(q + q^{-1})}{q^2 + q^{-2}}X \\ q^{-1}HY - qYH &= \frac{-2(q + q^{-1})}{q^2 + q^{-2}}Y \\ q^{-2}XY - q^2YX &= \frac{q^2 + q^{-2}}{2}H \end{aligned}$$

**Module structure, differentials:**

$$(f_j^i) = \begin{pmatrix} K^4 & 0 & 0 \\ \frac{2q^{-3/2}(q^{-1}-q)}{q^2+q^{-2}}K^3E & K^2 & 0 \\ \frac{2q^{-3/2}(q^{-1}-q)}{q^2+q^{-2}}FK^3 & 0 & K^2 \end{pmatrix} \quad \begin{aligned} d\omega_H &= -\frac{q^4+1}{2}\omega_X \wedge \omega_Y \\ d\omega_X &= \frac{-2(q^{-2}+1)}{q^2+q^{-2}}\omega_H \wedge \omega_X \\ d\omega_Y &= \frac{2(q^2+1)}{q^2+q^{-2}}\omega_H \wedge \omega_Y \end{aligned}$$

**Pairing:**

|     | $(u_1^1)^2$                          | $u_1^1u_2^1$ | $u_1^1u_1^2$ | $(u_2^1)^2$ | $u_2^1u_1^2$                     | $u_2^1u_2^2$ | $(u_1^2)^2$ | $u_1^2u_2^2$ | $(u_2^2)^2$                     |
|-----|--------------------------------------|--------------|--------------|-------------|----------------------------------|--------------|-------------|--------------|---------------------------------|
| $H$ | $\frac{2q^{-2}+2q^{-4}}{q^2+q^{-2}}$ | 0            | 0            | 0           | $\frac{2(q^{-1}-q)}{q^2+q^{-2}}$ | 0            | 0           | 0            | $\frac{-2q^4-2q^2}{q^2+q^{-2}}$ |
| $X$ | 0                                    | 1            | 0            | 0           | 0                                | $q$          | 0           | 0            | 0                               |
| $Y$ | 0                                    | 0            | 1            | 0           | 0                                | 0            | 0           | $q$          | 0                               |

**Right ideal  $\mathcal{R}_F$ :**  $u_1^1 + q^{-4}u_2^2 - (1 + q^{-4}), (u_2^1)^2, (u_1^2)^2, u_2^1u_1^2 + (q^3 - q)u_1^1, (u_1^1 - 1)u_2^1, (u_1^1 - 1)u_1^2$

**Left-invariant symmetric 2-forms:**

$$\begin{aligned} \omega_H \otimes \omega_H & & \omega_X \otimes \omega_X & & q^{-1}\omega_H \otimes \omega_X + q\omega_X \otimes \omega_H \\ q^2\omega_X \otimes \omega_Y + q^{-2}\omega_Y \otimes \omega_X & & \omega_Y \otimes \omega_Y & & q\omega_H \otimes \omega_Y + q^{-1}\omega_Y \otimes \omega_H \end{aligned}$$

**Braiding:**  $(1 - \sigma)(q^2 + \sigma) = 0$

$\ker(q^2 + \sigma) : q^{-2}\omega_H \otimes \omega_X - q^2\omega_X \otimes \omega_H, q^2\omega_H \otimes \omega_Y - q^{-2}\omega_Y \otimes \omega_H,$

$$q^3\omega_X \otimes \omega_Y - q^{-3}\omega_Y \otimes \omega_X - \frac{4(q-q^{-1})}{(q^2+q^{-2})^2}\omega_H \otimes \omega_H.$$

## 2. Quantum tangent space $\mathcal{X}_T$ :

$$H := \frac{2}{q^{-2} - q^2}(K^4 - 1), \quad X := q^{-5/2}FK^5, \quad Y := q^{-1/2}KE$$

**Real forms:**  $\mathcal{O}(\mathrm{SL}_q(2, \mathbb{R}))$

**Fund. repr.:**  $H = \frac{2}{q + q^{-1}} \begin{pmatrix} q^{-1} & 0 \\ 0 & -q \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

**Relations:**

$$\begin{aligned} q^2 HX - q^{-2}XH &= 2X \\ q^{-2}HY - q^2YH &= -2Y \\ q^{-3}XY - q^3YX + \frac{(q^2 + 1)^2(q^2 - 1)}{4q^3}H^2 &= \frac{q + q^{-1}}{2}H \end{aligned}$$

**Module structure, differentials:**

$$(f_j^i) = \begin{pmatrix} K^4 & \frac{q^2 - q^{-2}}{2}X & 0 \\ 0 & K^6 & 0 \\ 0 & 0 & K^2 \end{pmatrix} \quad \begin{aligned} d\omega_H &= \frac{-q^4 - q^2}{2}\omega_X \wedge \omega_Y \\ d\omega_X &= -2q^{-2}\omega_H \wedge \omega_Y \\ d\omega_Y &= 2q^2\omega_H \wedge \omega_X \end{aligned}$$

**Pairing:**

|     | $(u_1^1)^2$ | $u_1^1 u_2^1$ | $u_1^1 u_1^2$ | $(u_2^1)^2$ | $u_2^1 u_1^2$ | $u_2^1 u_2^2$ | $(u_1^2)^2$ | $u_1^2 u_2^2$ | $(u_2^2)^2$ |
|-----|-------------|---------------|---------------|-------------|---------------|---------------|-------------|---------------|-------------|
| $H$ | $2q^{-2}$   | 0             | 0             | 0           | 0             | 0             | 0           | 0             | $-2q^2$     |
| $X$ | 0           | $q^{-2}$      | 0             | 0           | 0             | $q^3$         | 0           | 0             | 0           |
| $Y$ | 0           | 0             | 1             | 0           | 0             | 0             | 0           | $q$           | 0           |

**Right ideal  $\mathcal{R}_T$ :**  $u_1^1 + q^{-2}u_2^2 - (1 + q^{-2}), (u_2^1)^2, (u_1^2)^2, u_2^1 u_1^2, (u_1^1 - q^{-2})u_2^1, (u_1^1 - 1)u_1^2$

**Left-invariant symmetric 2-forms:**  $p = (q^2 + 1)^2(q^2 - 1)/4$

$$\begin{aligned} \omega_H \otimes \omega_H - p\omega_X \otimes \omega_Y & & \omega_X \otimes \omega_X & & q^{-2}\omega_H \otimes \omega_X + q^2\omega_X \otimes \omega_H \\ q^3\omega_X \otimes \omega_Y + q^{-3}\omega_Y \otimes \omega_X & & \omega_Y \otimes \omega_Y & & q^2\omega_H \otimes \omega_Y + q^{-2}\omega_Y \otimes \omega_H \end{aligned}$$

**Braiding:** —

### 3. Quantum tangent space $\mathcal{X}_T$ :

$$H := \frac{2}{q^2 - q^{-2}}(K^{-4} - 1), \quad X := q^{3/2}FK^{-3}, \quad Y := q^{-1/2}KE$$

Real forms:  $\mathcal{O}(\mathrm{SL}_q(2, \mathbb{R}))$

Fund. repr.:  $H = \frac{2}{q + q^{-1}} \begin{pmatrix} q & 0 \\ 0 & -q^{-1} \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Relations:

$$\begin{aligned} q^{-2}HX - q^2XH &= 2X \\ q^2HY - q^{-2}YH &= -2Y \\ qXY - q^{-1}YX &= \frac{q + q^{-1}}{2}H \end{aligned}$$

Module structure, differentials:

$$(f_j^i) = \begin{pmatrix} K^{-4} & \frac{q^2 - q^{-2}}{2}X & 0 \\ 0 & K^{-2} & 0 \\ 0 & 0 & K^2 \end{pmatrix} \quad \begin{aligned} d\omega_H &= \frac{-1 - q^{-2}}{2}\omega_X \wedge \omega_Y \\ d\omega_X &= -2q^2\omega_H \wedge \omega_Y \\ d\omega_Y &= 2q^{-2}\omega_H \wedge \omega_X \end{aligned}$$

Pairing:

|     | $(u_1^1)^2$ | $u_1^1u_2^1$ | $u_1^1u_1^2$ | $(u_2^1)^2$ | $u_2^1u_1^2$ | $u_1^2u_2^2$ | $(u_1^2)^2$ | $u_1^2u_2^2$ | $(u_2^2)^2$ |
|-----|-------------|--------------|--------------|-------------|--------------|--------------|-------------|--------------|-------------|
| $H$ | $2q^2$      | 0            | 0            | 0           | 0            | 0            | 0           | 0            | $-2q^{-2}$  |
| $X$ | 0           | $q^2$        | 0            | 0           | 0            | $q^{-1}$     | 0           | 0            | 0           |
| $Y$ | 0           | 0            | 1            | 0           | 0            | 0            | 0           | $q$          | 0           |

Right ideal  $\mathcal{R}_T$ :  $u_1^1 + q^2u_2^2 - (1 + q^2), (u_2^1)^2, (u_1^2)^2, u_2^1u_1^2, (u_1^1 - q^2)u_2^1, (u_1^1 - 1)u_1^2$

Left-invariant symmetric 2-forms:

$$\begin{aligned} \omega_H \otimes \omega_H & & \omega_X \otimes \omega_X & & q^2\omega_H \otimes \omega_X + q^{-2}\omega_X \otimes \omega_H \\ q^{-1}\omega_X \otimes \omega_Y + q\omega_Y \otimes \omega_X & & \omega_Y \otimes \omega_Y & & q^{-2}\omega_H \otimes \omega_Y + q^2\omega_Y \otimes \omega_H \end{aligned}$$

Braiding:  $(1 - \sigma)(q^2 + \sigma) = 0$

$$\begin{aligned} \ker(q^2 + \sigma) : & q\omega_H \otimes \omega_X - q^{-1}\omega_X \otimes \omega_H, q^{-1}\omega_H \otimes \omega_Y - q\omega_Y \otimes \omega_H, \\ & \omega_X \otimes \omega_Y - \omega_Y \otimes \omega_X. \end{aligned}$$

#### 4. Quantum tangent space $\mathcal{X}_T$ :

$$H := \frac{2}{q - q^{-1}}(K^{-2} - 1), \quad X := q^{1/2}FK^{-1}, \quad Y := q^{1/2}K^{-1}E$$

**Real forms:**  $\mathcal{O}(\mathrm{SU}_q(2)), \mathcal{O}(\mathrm{SU}_q(1,1)), \mathcal{O}(\mathrm{SL}_q(2, \mathbb{R}))$

**Fund. repr.:**  $H = \frac{2}{q+1} \begin{pmatrix} q & 0 \\ 0 & -1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

**Relations:**

$$\begin{aligned} q^{-1}HX - qXH &= 2X \\ qHY - q^{-1}YH &= -2Y \\ qXY - q^{-1}YX - \frac{q - q^{-1}}{4}H^2 &= H \end{aligned}$$

**Module structure, differentials:**

$$(f_j^i) = \begin{pmatrix} K^{-2} & \frac{q-q^{-1}}{2}X & \frac{q-q^{-1}}{2}Y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{aligned} d\omega_H &= -q^{-1}\omega_X \wedge \omega_Y \\ d\omega_X &= -2q\omega_H \wedge \omega_X \\ d\omega_Y &= 2q^{-1}\omega_H \wedge \omega_Y \end{aligned}$$

**Pairing:**

|     | $(u_1^1)^2$ | $u_1^1u_2^1$ | $u_1^1u_1^2$ | $(u_2^1)^2$ | $u_2^1u_1^2$ | $u_2^1u_2^2$ | $(u_1^2)^2$ | $u_1^2u_2^2$ | $(u_2^2)^2$ |
|-----|-------------|--------------|--------------|-------------|--------------|--------------|-------------|--------------|-------------|
| $H$ | $2q$        | $0$          | $0$          | $0$         | $0$          | $0$          | $0$         | $0$          | $-2q^{-1}$  |
| $X$ | $0$         | $q$          | $0$          | $0$         | $0$          | $1$          | $0$         | $0$          | $0$         |
| $Y$ | $0$         | $0$          | $q$          | $0$         | $0$          | $0$          | $0$         | $1$          | $0$         |

**Right ideal  $\mathcal{R}_T$ :**  $u_1^1 + qu_2^2 - (1+q), (u_2^1)^2, (u_1^2)^2, u_2^1u_1^2, (u_1^1 - q)u_2^1, (u_1^1 - q)u_1^2$

**Left-invariant symmetric 2-forms:**

$$\begin{aligned} \omega_H \otimes \omega_H + \frac{1 - q^{-2}}{4}\omega_X \otimes \omega_Y & \quad \omega_X \otimes \omega_X & \quad q\omega_H \otimes \omega_X + q^{-1}\omega_X \otimes \omega_H \\ q^{-1}\omega_X \otimes \omega_Y + q\omega_Y \otimes \omega_X & \quad \omega_Y \otimes \omega_Y & \quad q^{-1}\omega_H \otimes \omega_Y + q\omega_Y \otimes \omega_H \end{aligned}$$

**Braiding:**  $(1 - \sigma)(q^2 + \sigma) = 0$

$$\ker(q^2 + \sigma) : \omega_H \otimes \omega_X - \omega_X \otimes \omega_H, \omega_H \otimes \omega_Y - \omega_Y \otimes \omega_H, \omega_X \otimes \omega_Y - \omega_Y \otimes \omega_X.$$

**5. Quantum tangent space  $\mathcal{X}_T$ :**

$$H := \frac{2}{q^{-1} - q}(\varepsilon_- K^{-2} - 1), \quad X := -q^{1/2} F \varepsilon_- K^{-1}, \quad Y := -q^{1/2} \varepsilon_- K^{-1} E$$

**Real forms:**  $\mathcal{O}(\mathrm{SU}_q(2)), \mathcal{O}(\mathrm{SU}_q(1,1)), \mathcal{O}(\mathrm{SL}_q(2, \mathbb{R}))$

**Fund. repr.:**  $H = \frac{2}{q-1} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

**Relations:**

$$\begin{aligned} q^{-1}HX - qXH &= -2X \\ qHY - q^{-1}YH &= 2Y \\ qXY - q^{-1}YX - \frac{q - q^{-1}}{4}H^2 &= -H \end{aligned}$$

**Module structure, differentials:**

$$(f_j^i) = \begin{pmatrix} \varepsilon_- K^{-2} & \frac{q^{-1}-q}{2}X & \frac{q^{-1}-q}{2}Y \\ 0 & \varepsilon_- & 0 \\ 0 & 0 & \varepsilon_- \end{pmatrix} \quad \begin{aligned} d\omega_H &= q^{-1}\omega_X \wedge \omega_Y \\ d\omega_X &= 2q\omega_H \wedge \omega_Y \\ d\omega_Y &= -2q^{-1}\omega_H \wedge \omega_X \end{aligned}$$

**Pairing:**

|     | $(u_1^1)^2$ | $u_1^1 u_2^1$ | $u_1^1 u_1^2$ | $(u_2^1)^2$ | $u_2^1 u_1^2$ | $u_2^1 u_2^2$ | $(u_1^2)^2$ | $u_1^2 u_2^2$ | $(u_2^2)^2$ |
|-----|-------------|---------------|---------------|-------------|---------------|---------------|-------------|---------------|-------------|
| $H$ | $-2q$       | $0$           | $0$           | $0$         | $0$           | $0$           | $0$         | $0$           | $2q^{-1}$   |
| $X$ | $0$         | $-q$          | $0$           | $0$         | $0$           | $-1$          | $0$         | $0$           | $0$         |
| $Y$ | $0$         | $0$           | $-q$          | $0$         | $0$           | $0$           | $0$         | $-1$          | $0$         |

**Right ideal  $\mathcal{R}_T$ :**  $u_1^1 - qu_2^2 - (1-q), (u_2^1)^2, (u_1^2)^2, u_2^1 u_1^2, (u_1^1 + q)u_2^1, (u_1^1 + q)u_1^2$

**Left-invariant symmetric 2-forms:**

$$\begin{aligned} \omega_H \otimes \omega_H + \frac{1 - q^{-2}}{4} \omega_X \otimes \omega_Y & \quad \omega_X \otimes \omega_X & \quad q\omega_H \otimes \omega_X + q^{-1}\omega_X \otimes \omega_H \\ q^{-1}\omega_X \otimes \omega_Y + q\omega_Y \otimes \omega_X & \quad \omega_Y \otimes \omega_Y & \quad q^{-1}\omega_H \otimes \omega_Y + q\omega_Y \otimes \omega_H \end{aligned}$$

**Braiding:**  $(1 - \sigma)(q^2 + \sigma) = 0$

$$\ker(q^2 + \sigma) : \omega_H \otimes \omega_X - \omega_X \otimes \omega_H, \omega_H \otimes \omega_Y - \omega_Y \otimes \omega_H, \omega_X \otimes \omega_Y - \omega_Y \otimes \omega_X.$$

## 6. Quantum tangent space $\mathcal{X}_F$ :

$$H := \frac{2}{q^2 - q^{-2}}(K^{-4} - 1), \quad X := q^{3/2}FK^{-3}, \quad Y := q^{3/2}K^{-3}E$$

**Real forms:**  $\mathcal{O}(\mathrm{SU}_q(2)), \mathcal{O}(\mathrm{SU}_q(1,1)), \mathcal{O}(\mathrm{SL}_q(2, \mathbb{R}))$

**Fund. repr.:**  $H = \frac{2}{q + q^{-1}} \begin{pmatrix} q & 0 \\ 0 & -q^{-1} \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

**Relations:**

$$\begin{aligned} q^{-2}HX - q^2XH &= 2X \\ q^2HY - q^{-2}YH &= -2Y \\ q^3XY - q^{-3}YX - \frac{(q^2 + 1)^2(q^2 - 1)}{4q^3}H^2 &= \frac{q + q^{-1}}{2}H \end{aligned}$$

**Module structure, differentials:**

$$(f_j^i) = \begin{pmatrix} K^{-4} & \frac{q^2 - q^{-2}}{2}X & \frac{q^2 - q^{-2}}{2}Y \\ 0 & K^{-2} & 0 \\ 0 & 0 & K^{-2} \end{pmatrix} \quad \begin{aligned} d\omega_H &= \frac{-q^{-2} - q^{-4}}{2}\omega_X \wedge \omega_Y \\ d\omega_X &= -2q^2\omega_H \wedge \omega_Y \\ d\omega_Y &= 2q^{-2}\omega_H \wedge \omega_X \end{aligned}$$

**Pairing:**

|     | $(u_1^1)^2$ | $u_1^1u_2^1$ | $u_1^1u_1^2$ | $(u_2^1)^2$ | $u_2^1u_1^2$ | $u_2^1u_2^2$ | $(u_1^2)^2$ | $u_1^2u_2^2$ | $(u_2^2)^2$ |
|-----|-------------|--------------|--------------|-------------|--------------|--------------|-------------|--------------|-------------|
| $H$ | $2q^2$      | 0            | 0            | 0           | 0            | 0            | 0           | 0            | $-2q^{-2}$  |
| $X$ | 0           | $q^2$        | 0            | 0           | 0            | $q^{-1}$     | 0           | 0            | 0           |
| $Y$ | 0           | 0            | $q^2$        | 0           | 0            | 0            | 0           | $q^{-1}$     | 0           |

**Right ideal  $\mathcal{R}_F$ :**  $u_1^1 + q^2u_2^2 - (1 + q^2), (u_2^1)^2, (u_1^2)^2, u_2^1u_1^2, (u_1^1 - q^2)u_1^2, (u_1^1 - q^2)u_1^2$

**Left-invariant symmetric 2-forms:**  $p = (1 + q^{-2})^2(1 - q^{-2})/4$

$$\begin{aligned} \omega_H \otimes \omega_H + p\omega_X \otimes \omega_Y & \quad \omega_X \otimes \omega_X & \quad q^2\omega_H \otimes \omega_X + q^{-2}\omega_X \otimes \omega_H \\ q^{-3}\omega_X \otimes \omega_Y + q^3\omega_Y \otimes \omega_X & \quad \omega_Y \otimes \omega_Y & \quad q^{-2}\omega_H \otimes \omega_Y + q^2\omega_Y \otimes \omega_H \end{aligned}$$

**Braiding:**  $(1 - \sigma)(q^2 + \sigma) = 0$

$$\begin{aligned} \ker(q^2 + \sigma) : q\omega_H \otimes \omega_X - q^{-1}\omega_X \otimes \omega_H, q^{-1}\omega_H \otimes \omega_Y - q\omega_Y \otimes \omega_H, \\ q^{-2}\omega_X \otimes \omega_Y - q^2\omega_Y \otimes \omega_X. \end{aligned}$$

**7. Quantum tangent space  $\mathcal{X}_T$ :**

$$H := \frac{2}{q^{-2} - q^2}(K^4 - 1), \quad X := q^{-1/2}FK, \quad Y := q^{-5/2}K^5E$$

**Real forms:**  $\mathcal{O}(\mathrm{SL}_q(2, \mathbb{R}))$

**Fund. repr.:**  $H = \frac{2}{q + q^{-1}} \begin{pmatrix} q^{-1} & 0 \\ 0 & -q \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

**Relations:**

$$\begin{aligned} q^2 HX - q^{-2}XH &= 2X \\ q^{-2}HY - q^2YH &= -2Y \\ q^{-3}XY - q^3YX + \frac{(q^2 + 1)^2(q^2 - 1)}{4q^3}H^2 &= \frac{q + q^{-1}}{2}H \end{aligned}$$

**Module structure, differentials:**

$$(f_j^i) = \begin{pmatrix} K^4 & 0 & \frac{q^{-2}-q^2}{2}Y \\ 0 & K^2 & 0 \\ 0 & 0 & K^6 \end{pmatrix} \quad \begin{aligned} d\omega_H &= \frac{-q^2-q^4}{2}\omega_X \wedge \omega_Y \\ d\omega_X &= -2q^{-2}\omega_H \wedge \omega_Y \\ d\omega_Y &= 2q^2\omega_H \wedge \omega_Y \end{aligned}$$

**Pairing:**

|     | $(u_1^1)^2$ | $u_1^1 u_2^1$ | $u_1^1 u_1^2$ | $(u_2^1)^2$ | $u_2^1 u_1^2$ | $u_2^1 u_2^2$ | $(u_1^2)^2$ | $u_1^2 u_2^2$ | $(u_2^2)^2$ |
|-----|-------------|---------------|---------------|-------------|---------------|---------------|-------------|---------------|-------------|
| $H$ | $2q^{-2}$   | 0             | 0             | 0           | 0             | 0             | 0           | 0             | $-2q^2$     |
| $X$ | 0           | 1             | 0             | 0           | 0             | $q$           | 0           | 0             | 0           |
| $Y$ | 0           | 0             | $q^{-2}$      | 0           | 0             | 0             | 0           | $q^3$         | 0           |

**Right ideal  $\mathcal{R}_T$ :**  $u_1^1 + q^{-2}u_2^2 - (1 + q^{-2}), (u_2^1)^2, (u_1^2)^2, u_2^1 u_1^2, (u_1^1 - 1)u_2^1, (u_1^1 - q^{-2})u_1^2$

**Left-invariant symmetric 2-forms:**  $p = (q^2 + 1)^2(q^2 - 1)/4$

$$\begin{aligned} \omega_H \otimes \omega_H - p\omega_X \otimes \omega_Y & & \omega_X \otimes \omega_X & & q^{-2}\omega_H \otimes \omega_X + q^2\omega_X \otimes \omega_H \\ q^3\omega_X \otimes \omega_Y + q^{-3}\omega_Y \otimes \omega_X & & \omega_Y \otimes \omega_Y & & q^2\omega_H \otimes \omega_Y + q^{-2}\omega_Y \otimes \omega_H \end{aligned}$$

**Braiding:** —

**8. Quantum tangent space  $\mathcal{X}_T$ :**

$$H := \frac{2}{q^2 - q^{-2}}(K^{-4} - 1), \quad X := q^{-1/2}FK, \quad Y := q^{3/2}K^{-3}E$$

**Real forms:**  $\mathcal{O}(\mathrm{SL}_q(2, \mathbb{R}))$

**Fund. repr.:**  $H = \frac{2}{q + q^{-1}} \begin{pmatrix} q & 0 \\ 0 & -q^{-1} \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

**Relations:**

$$\begin{aligned} q^{-2}HX - q^2XH &= 2X \\ q^2HY - q^{-2}YH &= -2Y \\ qXY - q^{-1}YX &= \frac{q + q^{-1}}{2}H \end{aligned}$$

**Module structure, differentials:**

$$(f_j^i) = \begin{pmatrix} K^{-4} & 0 & \frac{q^2 - q^{-2}}{2}Y \\ 0 & K^2 & 0 \\ 0 & 0 & K^{-2} \end{pmatrix} \quad \begin{aligned} d\omega_H &= \frac{-1 - q^{-2}}{2}\omega_X \wedge \omega_Y \\ d\omega_X &= -2q^2\omega_H \wedge \omega_X \\ d\omega_Y &= 2q^{-2}\omega_H \wedge \omega_Y \end{aligned}$$

**Pairing:**

|     | $(u_1^1)^2$ | $u_1^1 u_2^1$ | $u_1^1 u_1^2$ | $(u_2^1)^2$ | $u_2^1 u_1^2$ | $u_1^2 u_2^2$ | $(u_1^2)^2$ | $u_1^2 u_2^2$ | $(u_2^2)^2$ |
|-----|-------------|---------------|---------------|-------------|---------------|---------------|-------------|---------------|-------------|
| $H$ | $2q^2$      | 0             | 0             | 0           | 0             | 0             | 0           | 0             | $-2q^{-2}$  |
| $X$ | 0           | 1             | 0             | 0           | 0             | $q$           | 0           | 0             | 0           |
| $Y$ | 0           | 0             | $q^2$         | 0           | 0             | 0             | 0           | $q^{-1}$      | 0           |

**Right ideal  $\mathcal{R}_T$ :**  $u_1^1 + q^2 u_2^2 - (1 + q^2), (u_2^1)^2, (u_1^2)^2, u_2^1 u_1^2, (u_1^1 - 1)u_2^1, (u_1^1 - q^2)u_1^2$

**Left-invariant symmetric 2-forms:**

$$\begin{aligned} \omega_H \otimes \omega_H & & \omega_X \otimes \omega_X & & q^2 \omega_H \otimes \omega_X + q^{-2} \omega_X \otimes \omega_H \\ q^{-1} \omega_X \otimes \omega_Y + q \omega_Y \otimes \omega_X & & \omega_Y \otimes \omega_Y & & q^{-2} \omega_H \otimes \omega_Y + q^2 \omega_Y \otimes \omega_H \end{aligned}$$

**Braiding:**  $(1 - \sigma)(q^2 + \sigma) = 0$

$$\begin{aligned} \ker(q^2 + \sigma) : & q\omega_H \otimes \omega_X - q^{-1}\omega_X \otimes \omega_H, q^{-1}\omega_H \otimes \omega_Y - q\omega_Y \otimes \omega_H, \\ & \omega_X \otimes \omega_Y - \omega_Y \otimes \omega_X. \end{aligned}$$

**9. Quantum tangent space  $\mathcal{X}_T$ :**

$$H := \frac{2}{q^{-2} - q^2}(K^4 - 1), \quad X := q^{-1/2}FK, \quad Y := q^{-1/2}KE$$

**Real forms:**  $\mathcal{O}(\mathrm{SU}_q(2))$ ,  $\mathcal{O}(\mathrm{SU}_q(1, 1))$ ,  $\mathcal{O}(\mathrm{SL}_q(2, \mathbb{R}))$

**Fund. repr.:**  $H = \frac{2}{q + q^{-1}} \begin{pmatrix} q^{-1} & 0 \\ 0 & -q \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

**Relations:**

$$\begin{aligned} q^2 HX - q^{-2} XH &= 2X \\ q^{-2} HY - q^2 YH &= -2Y \\ q^{-1} XY - q YX &= \frac{q + q^{-1}}{2} H \end{aligned}$$

**Module structure, differentials:**

$$(f_j^i) = \begin{pmatrix} K^4 & 0 & 0 \\ 0 & K^2 & 0 \\ 0 & 0 & K^2 \end{pmatrix} \quad \begin{aligned} d\omega_H &= \frac{-1-q^2}{2} \omega_X \wedge \omega_Y \\ d\omega_X &= -2q^{-2} \omega_H \wedge \omega_Y \\ d\omega_Y &= 2q^2 \omega_H \wedge \omega_X \end{aligned}$$

**Pairing:**

|     | $(u_1^1)^2$ | $u_1^1 u_2^1$ | $u_1^1 u_1^2$ | $(u_2^1)^2$ | $u_2^1 u_1^2$ | $u_2^1 u_2^2$ | $(u_1^2)^2$ | $u_1^2 u_2^2$ | $(u_2^2)^2$ |
|-----|-------------|---------------|---------------|-------------|---------------|---------------|-------------|---------------|-------------|
| $H$ | $2q^{-2}$   | 0             | 0             | 0           | 0             | 0             | 0           | 0             | $-2q^2$     |
| $X$ | 0           | 1             | 0             | 0           | 0             | $q$           | 0           | 0             | 0           |
| $Y$ | 0           | 0             | 1             | 0           | 0             | 0             | 0           | $q$           | 0           |

**Right ideal  $\mathcal{R}_T$ :**  $u_1^1 + q^{-2}u_2^2 - (1 + q^{-2})$ ,  $(u_2^1)^2$ ,  $(u_1^2)^2$ ,  $u_2^1 u_1^2$ ,  $(u_1^1 - 1)u_2^1$ ,  $(u_1^1 - 1)u_1^2$

**Left-invariant symmetric 2-forms:**

$$\begin{aligned} \omega_H \otimes \omega_H & & \omega_X \otimes \omega_X & & q^{-2} \omega_H \otimes \omega_X + q^2 \omega_X \otimes \omega_H \\ q\omega_X \otimes \omega_Y + q^{-1}\omega_Y \otimes \omega_X & & \omega_Y \otimes \omega_Y & & q^2 \omega_H \otimes \omega_Y + q^{-2} \omega_Y \otimes \omega_H \end{aligned}$$

**Braiding:**  $(1 - \sigma)(q^2 + \sigma) = 0$

$$\begin{aligned} \ker(q^2 + \sigma) : & q^{-3} \omega_H \otimes \omega_X - q^3 \omega_X \otimes \omega_H, q^3 \omega_H \otimes \omega_Y - q^{-3} \omega_Y \otimes \omega_H, \\ & q^2 \omega_X \otimes \omega_Y - q^{-2} \omega_Y \otimes \omega_X. \end{aligned}$$

**10. Quantum tangent space  $\mathcal{X}_T$ :**

$$H := \frac{2}{q^{-1} - q}(K^2 - 1), \quad X := q^{-1/2}FK, \quad Y := q^{-1/2}KE$$

**Real forms:**  $\mathcal{O}(\mathrm{SU}_q(2))$ ,  $\mathcal{O}(\mathrm{SU}_q(1, 1))$ ,  $\mathcal{O}(\mathrm{SL}_q(2, \mathbb{R}))$

**Fund. repr.:**  $H = \frac{2}{q+1} \begin{pmatrix} 1 & 0 \\ 0 & -q \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

**Relations:**

$$\begin{aligned} qHX - q^{-1}XH &= 2X \\ q^{-1}HY - qYH &= -2Y \\ q^{-1}XY - qYX + \frac{q - q^{-1}}{4}H^2 &= H \end{aligned}$$

**Module structure, differentials:**

$$(f_j^i) = \begin{pmatrix} K^2 & 0 & 0 \\ 0 & K^2 & 0 \\ 0 & 0 & K^2 \end{pmatrix} \quad \begin{aligned} d\omega_H &= -q\omega_X \wedge \omega_Y \\ d\omega_X &= -2q^{-1}\omega_H \wedge \omega_X \\ d\omega_Y &= 2q\omega_H \wedge \omega_Y \end{aligned}$$

**Pairing:**

|     | $(u_1^1)^2$ | $u_1^1 u_2^1$ | $u_1^1 u_1^2$ | $(u_2^1)^2$ | $u_2^1 u_1^2$ | $u_2^1 u_2^2$ | $(u_1^2)^2$ | $u_1^2 u_2^2$ | $(u_2^2)^2$ |
|-----|-------------|---------------|---------------|-------------|---------------|---------------|-------------|---------------|-------------|
| $H$ | $2q^{-1}$   | 0             | 0             | 0           | 0             | 0             | 0           | 0             | $-2q$       |
| $X$ | 0           | 1             | 0             | 0           | 0             | $q$           | 0           | 0             | 0           |
| $Y$ | 0           | 0             | 1             | 0           | 0             | 0             | 0           | $q$           | 0           |

**Right ideal  $\mathcal{R}_T$ :**  $u_1^1 + q^{-1}u_2^2 - (1 + q^{-1})$ ,  $(u_2^1)^2$ ,  $(u_1^2)^2$ ,  $u_2^1 u_1^2$ ,  $(u_1^1 - 1)u_2^1$ ,  $(u_1^1 - 1)u_1^2$

**Left-invariant symmetric 2-forms:**

$$\begin{aligned} \omega_H \otimes \omega_H + \frac{1 - q^2}{4} \omega_X \otimes \omega_Y & \quad \omega_X \otimes \omega_X & \quad q^{-1} \omega_H \otimes \omega_X + q \omega_X \otimes \omega_H \\ q \omega_X \otimes \omega_Y + q^{-1} \omega_Y \otimes \omega_X & \quad \omega_Y \otimes \omega_Y & \quad q \omega_H \otimes \omega_Y + q^{-1} \omega_Y \otimes \omega_H \end{aligned}$$

**Braiding:**  $(1 - \sigma)(q^2 + \sigma) = 0$

$\ker(q^2 + \sigma) : \omega_H \otimes \omega_X - \omega_X \otimes \omega_H, \omega_H \otimes \omega_Y - \omega_Y \otimes \omega_H, \omega_X \otimes \omega_Y - \omega_Y \otimes \omega_X.$

### 11. Quantum tangent space $\mathcal{X}_T$ :

$$H := \frac{2}{q - q^{-1}}(\varepsilon_- K^2 - 1), \quad X := q^{-1/2} F K, \quad Y := q^{-1/2} K E$$

**Real forms:**  $\mathcal{O}(\mathrm{SU}_q(2))$ ,  $\mathcal{O}(\mathrm{SU}_q(1, 1))$ ,  $\mathcal{O}(\mathrm{SL}_q(2, \mathbb{R}))$

**Fund. repr.:**  $H = \frac{2}{1 - q} \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

**Relations:**

$$\begin{aligned} qHX - q^{-1}XH &= -2X \\ q^{-1}HY - qYH &= 2Y \\ q^{-1}XY - qYX + \frac{q - q^{-1}}{4}H^2 &= -H \end{aligned}$$

**Module structure, differentials:**

$$(f_j^i) = \begin{pmatrix} \varepsilon_- K^2 & 0 & 0 \\ 0 & K^2 & 0 \\ 0 & 0 & K^2 \end{pmatrix} \quad \begin{aligned} d\omega_H &= q\omega_X \wedge \omega_Y \\ d\omega_X &= 2q^{-1}\omega_H \wedge \omega_X \\ d\omega_Y &= -2q\omega_H \wedge \omega_Y \end{aligned}$$

**Pairing:**

|     | $(u_1^1)^2$ | $u_1^1 u_2^1$ | $u_1^1 u_1^2$ | $(u_2^1)^2$ | $u_2^1 u_1^2$ | $u_2^1 u_2^2$ | $(u_1^2)^2$ | $u_1^2 u_2^2$ | $(u_2^2)^2$ |
|-----|-------------|---------------|---------------|-------------|---------------|---------------|-------------|---------------|-------------|
| $H$ | $-2q^{-1}$  | 0             | 0             | 0           | 0             | 0             | 0           | 0             | $2q$        |
| $X$ | 0           | 1             | 0             | 0           | 0             | $q$           | 0           | 0             | 0           |
| $Y$ | 0           | 0             | 1             | 0           | 0             | 0             | 0           | $q$           | 0           |

**Right ideal  $\mathcal{R}_T$ :**  $u_1^1 - q^{-1}u_2^2 - (1 - q^{-1})$ ,  $(u_2^1)^2$ ,  $(u_1^2)^2$ ,  $u_2^1 u_1^2$ ,  $(u_1^1 - 1)u_2^1$ ,  $(u_1^1 - 1)u_1^2$

**Left-invariant symmetric 2-forms:**

$$\begin{aligned} \omega_H \otimes \omega_H + \frac{1 - q^2}{4} \omega_X \otimes \omega_Y & \quad \omega_X \otimes \omega_X & \quad q^{-1} \omega_H \otimes \omega_X + q \omega_X \otimes \omega_H \\ q \omega_X \otimes \omega_Y + q^{-1} \omega_Y \otimes \omega_X & \quad \omega_Y \otimes \omega_Y & \quad q \omega_H \otimes \omega_Y + q^{-1} \omega_Y \otimes \omega_H \end{aligned}$$

**Braiding:**  $(1 - \sigma)(q^2 + \sigma) = 0$

$\ker(q^2 + \sigma) : \omega_H \otimes \omega_X - \omega_X \otimes \omega_H, \omega_H \otimes \omega_Y - \omega_Y \otimes \omega_H, \omega_X \otimes \omega_Y - \omega_Y \otimes \omega_X.$

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